Global Inverse Spectral Problems for Rational Matrix Functions

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ABSTRACT

Given rational matrix functions $\Psi_1(\lambda) = I_m + C_1(\lambda I_{n_1} - A_1)^{-1}B_1$ and $\Psi_2(\lambda) = I_m + C_2(\lambda I_{n_2} - A_2)^{-1}B_2$ which are analytic and invertible on the unit circle, we characterize in terms of the operators $A_1, B_1, C_1, A_2, B_2, C_2$ when there exists a single rational matrix function $W(\lambda) = I_m + C(\lambda I_n - A)^{-1}B$ such that $WH_m^{2\perp} = \Psi_1 H_m^{2\perp}$ and $WH_m^2 = \Psi_2 H_m^2$. When this is the case, we give explicit formulae for A, B, C in terms of $A_1, B_1, C_1, A_2, B_2, C_2$. Applications include Wiener-Hopf factorization, J-inner-outer factorization, and coprime factorization. The results on J-inner-outer factorization have application to a model reduction problem for discrete time linear systems.

INTRODUCTION

Let Y be a finite dimensional Hilbert space. If $m = \dim Y$, we identify $m \times m$ matrices as operators on Y. We let L_Y^2 be the space of norm

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square-integrable Y-valued functions on the unit circle $\{\lambda \mid |\lambda| = 1\}$. The space H_Y^2 is the usual Hardy subspace of L_Y^2, the subspace of functions which are radial limits of functions analytic on the unit disk $\mathscr{D} = \{\lambda \mid |\lambda| < 1\}$; $H_Y^{2\perp}$ denotes the orthogonal complement of H_Y^2 in L_Y^2 and consists of boundary values of functions analytic on the exterior $\mathscr{D}_e = \{\lambda \mid |\lambda| > 1\}$ of the unit disk with value 0 at ∞. The main result from [2], reduced here to the rational case for simplicity, can be stated as follows: if $M^x = \Psi_1 H_Y^{2\perp}$ and $M = \Psi_2 H_Y^2$ for rational matrix functions Ψ_1 and Ψ_2 with no poles or zeros on the unit circle, there exists a single matrix function Θ such that $M^x = \Theta H_Y^{2\perp}$ and $M = \Theta H_Y^2$ if and only if the direct sum condition

$$L_Y^2 = M^x \dot{+} M$$

holds. This result was shown to have various applications to factorization; indeed, if $\Psi_1 = I_Y$ and $\Psi_2 = W$, then $W = \Theta \cdot \Theta^{-1} W$ is a canonical Wiener-Hopf factorization of W, while if $\Psi_1(\lambda) = W(\bar{\lambda}^{-1})^{*-1}$ and $\Psi_2(\lambda) = W(\lambda)$, then Θ is essentially the inner factor of W. Computation of the representor Θ was shown to amount to a computation of a basis for the "wandering subspace" $\mathscr{L} := \lambda M^x \cap M$, in imitation of Halmos's original proof of the Beurling-Lax theorem [10].

A different approach to factorization, motivated by the cascade connection of systems, is taken in the book [6]. One assumes one has a realization $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1} B$ for the rational matrix function W as the transfer function of a system; here $\dim X < \infty$ is the McMillan degree of W if one assumes that the realization is minimal. Then minimal factorizations of W are analyzed in terms of the geometry of certain invariant subspaces of the state operator A in the state space X; shift-invariant subspaces of L_Y^2 never appear.

In this paper we synthesize these two points of view by showing how the shift-invariant subspaces $WH_Y^{2\perp}$ and WH_Y^2 can be described explicitly from the objects in a state space realization $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1} B$ for W, or equivalently, in terms of certain "spectral data" of W. This leads us to the notion of a *canonical set of spectral data* (c.s.s.d.) on $\mathscr{D}$ for W, a collection of operators $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ closely associated with realizations for W and for W^{-1}, from which the space $M = WH_Y^2$ can be determined explicitly; conversely, a c.s.s.d. on $\mathscr{D}$ for W can be determined from knowledge of the subspace $M = WH_Y^2$. A similar connection is shown to exist between a so-called *canonical set of spectral data* on $\mathscr{D}_e$ for W, $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^x\}$, and the subspace $M^x = WH_Y^{2\perp}$. This is the direct analysis covered in Section 1.

In Section 2 we solve the inverse problem: given a collection of operators $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$, determine when there is a rational matrix function W for which this set is a c.s.s.d. on $\mathscr{D}$ for W. The idea is simply to write down the subspace $M \subset L^2_Y$, which must be WH^2_Y if W is to be a solution of the inverse problem, determine what it means for M to be shift-invariant (i.e. $\lambda M \subset M$), and then quote the Beurling-Lax theorem to deduce that indeed $M = WH^2_Y$ for some function W; the shift invariance of M is equivalent to $\hat{T}$ satisfying a Lyapunov equation. Then it is a simple matter to verify that the original data must be a c.s.s.d. on $\mathscr{D}$ for this W. A c.s.s.d. on $\mathscr{D}_e$ can be characterized in a similar way.

In Section 3 we solve the more stringent inverse spectral problem: given two collections of operators $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ and $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^{x}\}$, when is the first a c.s.s.d. on $\mathscr{D}$ and the second a c.s.s.d. on $\mathscr{D}_e$ for the same function W? Necessary conditions from the analysis in Section 2 are that $\hat{T}$ and $\hat{T}^{x}$ satisfy the appropriate Lyapunov equations. This enables one to write down subspaces M^{x} and M in L^2_Y; the problem is to produce a rational matrix function W such that $M^{x} = WH^{2\perp}_Y$ and $M = WH^2_Y$. One uses the invariant subspace theorem from [2] instead of the Beurling-Lax theorem: the extra condition required is that $L^2_Y = M^{x} \dot{+} M$. By direct computation this is shown to be equivalent to the invertibility of the operator $\hat{\mathscr{T}}$ given by

$$\hat{\mathscr{T}} = \begin{pmatrix} -\hat{T}^{x} & -\sum_{j=1}^{\infty} A_{z-}^{-j} B_- C_+ A_{p+}^{j-1} \\ \sum_{j=1}^{\infty} A_{z+}^{j-1} B_+ C_- A_{p-}^{-j} & \hat{T} \end{pmatrix}.$$

(The operators A_{z+} and A_{p+} have spectrum in $\mathscr{D}$, while A_{z-} and A_{p-} have spectrum in $\mathscr{D}_e$, so the infinite sums are convergent.) Moreover, one can borrow a formula from [9] which describes solutions of a related inverse spectral problem, to write down a realization for the desired function $W(\lambda)$. Indeed, in this way one can bypass the use of the theorem from [2] by checking directly that this formula provides the solution; there results a new "state space" proof of the main result of [2], at least for the rational case.

In Section 4 we show how the problems of canonical Wiener-Hopf factorization, J-inner-outer factorization, and left or right coprime factorization of a given function $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B$ can be set up as inverse spectral problems in the sense of Section 3. In particular the result on J-inner-outer factorization gives an alternative derivation of the main result of our earlier report [3] on model reduction for discrete time linear systems.

In a paper parallel to this one [5], we obtained a local form for the results of this paper. There we give a local description of the subspaces $M = WH^2_Y$

and $M^{\times} = WH_Y^{2\perp}$ from a knowledge of all the left zero chains for $W(\lambda)$ and right zero chains for $W^{-1}(\lambda)$ [or, by definition, right pole chains for $W(\lambda)$], and solve inverse problems analogous to those in this paper. It is easy to see how to form a c.s.s.d. on $\mathscr{D}$ and $\mathscr{D}_e$ for W from a knowledge of all the left zero and right pole chains of W, so a number of results from [5] can be derived by plugging into the results of this paper. Nevertheless we feel that [5] has its own special insights and that the explicit formulas there are better obtained by the self-contained direct local approach there.

Finally we should mention that this paper has close connection with the paper [9]. Indeed, if $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. on $\mathscr{D}$ and $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^{\times}\}$ is a c.s.s.d. on $\mathscr{D}$ for W, then

$$\left((C_-, C_+), \begin{pmatrix} A_{p-} & 0 \\ 0 & A_{p+} \end{pmatrix}\right)$$

is a left pole pair for W^{-1} and

$$\left(\begin{pmatrix} B_- \\ B_+ \end{pmatrix}, \begin{pmatrix} A_{z-} & 0 \\ 0 & A_{z+} \end{pmatrix}\right)$$

is a right zero pair for W^{-1}; these are the objects for the inverse spectral problem considered in [9]. Also, as mentioned above, the formula found there to describe the set of solutions of this inverse spectral problem suggested the explicit formula for the solution of ours. Our contribution here is to identify the extra piece of spectral information (i.e. $\hat{T}$ and $\hat{T}^{\times}$) required about W to determine the invariant subspaces $M^{\times} = WH_Y^{2\perp}$ and $M = WH_Y^2$, and to show how various factorization problems can be set up as special instances of this more constrained inverse spectral problem.

1. DIRECT ANALYSIS

Let $W(\lambda)$ be a rational matrix valued function analytic and invertible on the unit circle $\{\lambda \mid |\lambda| = 1\}$ and at ∞. The values $W(\lambda)$ of W we think of as operators on a finite dimensional Hilbert space Y; thus $Y \simeq \mathbb{C}^m$ for some m. Without loss of generality we assume that $W(\infty) = I_Y$. In that case $W(\lambda)$ admits a minimal realization

$$W(\lambda) = I_Y + C(\lambda I_{X_p} - A_p)^{-1}\tilde{B}, \qquad \lambda \in \rho(A_p), \tag{1.1}$$

where $A_p\colon X_p \to X_p$, $C\colon X_p \to Y$, $\tilde{B}\colon Y \to X_p$ are bounded linear operators such that

$$\operatorname{Ker}\operatorname{col}\left(CA_p^j\right)_{j=0}^{\infty} = (0), \tag{1.2}$$

$$\operatorname{Im}\operatorname{row}\left(A_p^j\tilde{B}\right)_{j=0}^{\infty} = X_p. \tag{1.2$'$}$$

Here $\dim X_p < \infty$ and is equal to the McMillan degree of W (see e.g. [6]). We assume that X_p is also endowed with an inner product making it a Hilbert space. Whenever (C, A_p) is a pair of operators satisfying (1.2) [i.e., (C, A_p) is *observable*] for which there exists an operator $\tilde{B}$ such that (1.1) and (1.2′) hold, we say that (C, A_p) is a *right pole pair* for W. Right pole pairs are closely related to right zero chains (or equivalently right root functions) for the inverse function $W(\lambda)^{-1}$ (see [9]); this is the reason for the terminology *right* pole pair. The function $W^{-1}(\lambda) = W(\lambda)^{-1}$ is of the same type as $W(\lambda)$ and hence also has a realization

$$W(\lambda)^{-1} = I_Y - \tilde{C}\left(\lambda I_{X_z} - A_z\right)^{-1}B \tag{1.3}$$

such that

$$\operatorname{Im}\operatorname{row}\left(A_z^jB\right)_{j=0}^{\infty} = X_z, \tag{1.4}$$

$$\operatorname{Ker}\operatorname{col}\left(\tilde{C}A_z^j\right)_{j=0}^{\infty} = (0). \tag{1.4$'$}$$

Indeed, one way to do this is to take $X_z = X_p$, $\tilde{C} = C$, $A_z = A_p - \tilde{B}C$, and $B = \tilde{B}$. It is always the case that $\dim X_z = \dim X_p =$ the McMillan degree of W. Whenever (A_z, B) is a pair of operators satisfying (1.4) [i.e., (A_z, B) is *controllable*] such that there is a $\tilde{C}$ for which (1.3) and (1.4′) hold, we say that (A_z, B) is a *left zero pair* for W. Left zero pairs are closely connected with left zero chains (or left root functions) for the function W (see again [9]). Since both $W(\lambda)$ and $W^{-1}(\lambda)$ are analytic on the unit circle, by our observability and controllability assumptions necessarily both A_z and A_p have no spectrum on the unit circle whenever (C, A_p) is a right pole pair and (A_z, B) is a left zero pair for $W(\lambda)$.

Our aim in this section is to describe the spaces $M := WH_Y^2$ and $M^\times := WH_Y^{2\perp}$ as subspaces of L_Y^2 directly in terms of a given right pole pair (C, A_p) and left zero pair (A_z, B) for W. The description will actually be of their images under the discrete Fourier transform. Note that L_Y^2 has an orthogonal direct sum decomposition $L_Y^2 = H_Y^{2\perp} \oplus H_Y^2$. Any element of H_Y^2 has the form

$f(\lambda)=\sum_{n=1}^{\infty} f_n \lambda^{n-1}$ where $\operatorname{col}(f_n)_{n=1}^{\infty} \in l_Y^2$. Let $\mathscr{F}_+ : H_Y^2 \to l_Y^2$ be the identification map $\mathscr{F}_+ : f(\lambda) \to \operatorname{col}(f_n)_{n=1}^{\infty}$. Similarly, an element g of $H_Y^{2\perp}$ has the form $g(\lambda)=\sum_{n=1}^{\infty} g_n \lambda^{-n}$, where $\operatorname{col}(g_n)_{n=1}^{\infty} \in l_Y^2$. Let $\mathscr{F}_- : H_Y^{2\perp} \to l_Y^2$ be the identification map $\mathscr{F}_- : g(\lambda) \to \operatorname{col}(g_n)_{n=1}^{\infty}$. Then the map $\mathscr{F} : \mathscr{F}_- \oplus \mathscr{F}_+ : L_Y^2 \simeq H_Y^{2\perp} \oplus H_Y^2 \to l_Y^2 \oplus l_Y^2$ identifies L_Y^2 with the direct sum space $l_Y^2 \oplus l_Y^2$. Our goal then is to describe $\mathscr{F}M = \mathscr{F}(WH_Y^2)$ and $\mathscr{F}M^x = \mathscr{F}(WH_Y^{2\perp})$ explicitly in terms of a given right pole pair (C, A_p) and left zero pair (A_z, B) to the extent possible. By assumption neither A_z nor A_p has spectrum on the unit circle. Let $P_p = (1/2\pi i)\int_{|\lambda|=1}(\lambda I_{X_p} - A_p)^{-1} d\lambda$ be the Riesz projection of A_p for the piece of the spectrum of A_p inside the unit disk $\mathscr{D} = \{\lambda \mid |\lambda| < 1\}$, and let $P_z = (1/2\pi i)\int_{|\lambda|=1}(\lambda I_{X_z} - A_z)^{-1} d\lambda$ be the corresponding object for A_z. Let us say that the pair (C_+, A_{p+}) (where $C_+ : X_{p+} \to Y$, $A_{p+} : X_{p+} \to X_{p+}$) is a *right pole pair* for W on $\mathscr{D}$ if $X_{p+} = \operatorname{Im} P_p$, $C_+ = C|\operatorname{Im} P_p$, and $A_{p+} = A_p|\operatorname{Im} P_p$ for a right pole pair (C, A_p) for W. Similarly, we say that (A_{z+}, B_+) (where $A_{z+} : X_{z+} \to X_{z+}$, $B_+ : Y \to X_{z+}$) is a *left zero pair* for W on $\mathscr{D}$ if $A_{z+} = A_z|\operatorname{Im} P_z$ and $B_+ = P_z B$ for a left zero pair (A_z, B) for W. It turns out that knowledge of a right pole pair (C_+, A_{p+}) and left zero pair (A_{z+}, B_+) for W on $\mathscr{D}$ does not always completely determine the subspace $M = WH_Y^2$, but does give a lot of information; namely, knowledge of a right pole pair (C_+, A_{p+}) on $\mathscr{D}$ specifies the subspace $M_p := P_{H_Y^{2\perp}} M$, and knowledge of a left zero pair (A_{z+}, B_+) on $\mathscr{D}$ specifies the subspace $M_z := M \cap H_Y^2$. The result is as follows.

THEOREM 1.1. *Suppose (C_+, A_{p+}) is a right pole pair for the rational matrix function W on $\mathscr{D}$, and (A_{z+}, B_+) is a left zero pair for W on $\mathscr{D}$. Then if $M_p = P_{H_y^{2\perp}}(WH_Y^2)$ and $M_z = WH_Y^2 \cap H_Y^2$, we have*

$$\mathscr{F}_- M_p = \operatorname{Im}\operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty} \tag{1.5}$$

and

$$\mathscr{F}_+ M_z = \operatorname{Ker}\operatorname{row}\left(A_{z+}^{n-1} B_+\right)_{n=1}^{\infty}. \tag{1.6}$$

Proof. Assume that $C_+ = C|\operatorname{Im} P_p$, $A_{p+} = A_p|\operatorname{Im} P_p$, $A_{z+} = A_z|\operatorname{Im} P_z$, $B_+ = P_z B$, where (C, A_p) is a right pole pair for W, (A_z, B) is a left zero pair for W, and P_p and P_z are the Riesz projections for A_p and A_z respectively for the parts of the spectrum in $\mathscr{D}$. Let $W(\lambda) = \sum_{n=-\infty}^{\infty} T_n \lambda^n$ be the Laurent expansion for W, valid for $|\lambda| = 1$. Then we obtain from (1.1)

$$T_n = \begin{cases} CA_{p+}^{-n-1}P_p\tilde{B}, & n = -1, -2, \dots, \\ I_Y - CA_{p-}^{-1}(I - P_p)\tilde{B}, & n = 0, \\ -CA_{p-}^{-n-1}(I - P_p)\tilde{B}, & n = 1, 2, \dots, \end{cases} \tag{1.7}$$

where we have set $A_{p+} := A_p|\operatorname{Im} P_p$ and $A_{p-} := A_p|\operatorname{Im}(I - P_p)$. First we compute $M_p := P_{H_Y^{2\perp}}M$ and then $M_z := M \cap H_Y^2$. Choose $f(\lambda) = \sum_{n=1}^{\infty} f_n \lambda^{n-1}$ in H_Y^2. Then $P_{H_Y^{2\perp}}Wf$ is given by

$$\left(P_{H_Y^{2\perp}}Wf\right)(\lambda) = \sum_{j=1}^{\infty} g_j \lambda^j,$$

where

$$g_j = \sum_{n=1}^{\infty} CA_{p+}^{j+n-2}P_p\tilde{B}f_n = \sum_{n=1}^{\infty} CA_{p+}^{j-1}P_pA_p^{n-1}\tilde{B}f_n.$$

Then

$$\operatorname{col}(g_j)_{j=1}^{\infty} = \operatorname{col}\left(CA_{p+}^{j-1}\right)_{j=1}^{\infty} \cdot \operatorname{row}\left(P_pA_p^{n-1}\tilde{B}\right)_{n=1}^{\infty} \cdot \operatorname{col}(f_n)_{n=1}^{\infty}. \tag{1.8}$$

By our controllability assumption (1.2′) we see from (1.6) that

$$\mathscr{F}_- M_p = \mathscr{F}_-\left(P_{H_Y^{2\perp}}M\right) = \operatorname{Im}\operatorname{col}\left(CA_{p+}^{j-1}\right)_{j=1}^{\infty}.$$

This verifies (1.5).

To compute $M \cap H_Y^2$, we compute the Laurent series for $W^{-1}(\lambda)$ using (1.3). One gets

$$W^{-1}(\lambda) = \sum_{n=-\infty}^{\infty} T_n^{x}\lambda^n \qquad \text{for} \quad |\lambda| = 1,$$

where

$$T_n^{x} = \begin{cases} -\tilde{C}A_{z+}^{-n-1}P_zB, & n = -1, -2, \dots, \\ I_Y + \tilde{C}A_{z-}^{-1}(I - P_z)B, & n = 0, \\ \tilde{C}A_{z-}^{-n-1}(I - P_z)B, & n = 1, 2, \dots, \end{cases} \tag{1.9}$$

where we have set $A_{z+} := A_z|\operatorname{Im} P_z$ and $A_{z-} := A_z|\operatorname{Im}(I - P_z)$. If $g \in M \cap H_Y^2 = WH_Y^2 \cap H_Y^2$, then g and $W^{-1}g$ are both in H_Y^2, and conversely. So suppose $g(\lambda) = \sum_{j=1}^{\infty} g_j \lambda^{j-1} \in H_Y^2$. Then $W^{-1}g \in H_Y^2$ if and only if $\sum_{n=1}^{\infty} \tilde{C} A_z^{j-1} A_{z+}^{n-1} P_z B g_n = 0$, $j = 1,2,\ldots$, i.e. if and only if

$$0 = \operatorname{col}\left(\tilde{C} A_z^{j-1}\right)_{j=1}^{\infty} \cdot \operatorname{row}\left(A_{z+}^{n-1} B_+\right)_{n=1}^{\infty} \cdot \operatorname{col}\left(g_n\right)_{n=1}^{\infty}.$$

By the observability condition (1.4′) the identity (1.6) follows. ∎

The reader should note that the subspace $P_{H_Y^{2\perp}} M$ described by (1.5) is finite dimensional, as well as $H_Y^2 \ominus M_z$ where M_z is described by (1.6), since we are assuming that X_{z+} and X_{p+} are finite dimensional.

In order to describe the subspace WH_Y^2 itself, we must know how M is determined by M_p and M_z. By general properties of linear spaces, one easily sees that there is a one-to-one correspondence between subspaces $M \subset L_Y^2$ with the associated subspaces $M_p = P_{H_Y^{2\perp}} M$ and $M_z = M \cap H_Y^2$ specified and fixed on the one hand, and linear operators T from $\mathscr{F}_- M_p$ into $l_Y^2 \ominus \mathscr{F}_+ M_z$ on the other. The correspondence is given by

$$\mathscr{F} M = \left\{ f \oplus (Tf + g) : f \in \mathscr{F}_- M_p,\ g \in \mathscr{F}_+ M_z \right\}. \tag{1.10}$$

Let us now suppose that (C_+, A_{p+}) is a right pole pair on $\mathscr{D}$ and (A_{z+}, B_+) is a left zero pair on $\mathscr{D}$ for some known rational matrix function W, and let $M = WH_Y^2$. Then there must be an operator $T: \mathscr{F}_- M_p \to l_Y^2 \ominus \mathscr{F}_+ M_z$ such that $\mathscr{F} M$ is as in (1.10). Define the operator $\hat{T}: X_{p+} \to X_{z+}$ by

$$\hat{T} = \operatorname{row}\left(A_{z+}^{n-1} B_+\right)_{n=1}^{\infty} \cdot T \cdot \operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty}. \tag{1.11}$$

By the characterizations of $\mathscr{F}_- M_p$ and $\mathscr{F}_+ M_z$ given by (1.5) and (1.6), we see that conversely, given $\hat{T}: X_{p+} \to X_{z+}$, there is a unique $T: \mathscr{F}_- M_p \to l_Y^2 \ominus \mathscr{F}_+ M_z$ such that (1.11) is satisfied; this knowledge of the triple $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ completely specifies the subspace $M = WH_Y^2$. For this reason we call any triple $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ associated in this way with a rational matrix function W a *canonical set of spectral data* (c.s.s.d.) for W on $\mathscr{D}$. We will give a characterization of the "coupling operator" $\hat{T}$ which is more intrinsic to the function $W(\lambda)$ below (see Remark 1.3).

We summarize the analysis thus far as follows. The content beyond that of Theorem 1.1 is true just by definition.

THEOREM 1.2. *Suppose* $\{(C_+, A_{p+}),(A_{z+}, B_+),\hat{T}\}$ *is a c.s.s.d. for* W *on* $\mathscr{D}$. *Then* $M = WH_Y^2$ *is given by*

$$\mathscr{F}M = \left\{ f \oplus (Tf + g) : f \in \mathscr{F}_- M_p,\ g \in \mathscr{F}_+ M_z \right\},$$

where

$$\mathscr{F}_- M_p = \operatorname{Im}\operatorname{col}\left(C_+ A_{p+}^{j-1} \right)_{j=1}^{\infty},$$

$$\mathscr{F}_+ M_z = \operatorname{Ker}\operatorname{row}\left(A_{z+}^{n-1} B_+ \right)_{n=1}^{\infty},$$

and $T : \mathscr{F}_- M_p \to l_Y^2 \ominus \mathscr{F}_+ M_z$ *is the unique solution of*

$$\hat{T} = \operatorname{row}\left(A_{z+}^{n-1} B_+ \right)_{n=1}^{\infty} \cdot T \cdot \operatorname{col}\left(C_+ A_{p+}^{j-1} \right)_{j=1}^{\infty}.$$

REMARK 1.2. If (C_+, A_{p+}) is a right pole pair for a function W on $\mathscr{D}$, where $C_+ : X_{p+} \to Y$ and $A_{p+} : X_{p+} \to X_{p+}$, and if $S_1 : X'_{p+} \to X_{p+}$ is a linear bijection, then it is easy to see that $(C_+ S_1, S_1^{-1} A_{p+} S_1)$ is also a right pole pair for W on $\mathscr{D}$. Similarly, if (A_{z+}, B_+) is a left zero pair for W on $\mathscr{D}$, where $A_{z+} : X_{z+} \to X_{z+}$ and $B_+ : Y \to X_{z+}$, and if $S_2 : X_{z+} \to X'_{z+}$ is a linear bijection, then $(S_2 A_{z+} S_2^{-1}, S_2 B_+)$ is also a left zero pair for W on $\mathscr{D}$. In fact any pair of pole pairs and zero pairs are related in this way. It is useful to know how the associated coupling operator $\hat{T}$ for W transforms when one changes the pole pair and/or the zero pair in this way. It is straightforward to verify that $\{(C_+ S_1, S_1^{-1} A_{p+} S_1), (S_2 A_{z+} S_2^{-1}, S_2 B_+), S_2 \hat{T} S_1\}$ is a c.s.s.d. for W on $\mathscr{D}$ whenever $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is.

We next compute $\hat{T}$ for a right pole pair and left zero pair for W on $\mathscr{D}$ of a special form.

THEOREM 1.3. *Suppose* $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1} B$ *is a minimal realization for the rational matrix function* W *which is analytic and invertible on the unit circle. Set* $A^x := A - BC$, *and let* P *and* P^x *denote the Riesz projections for* A *and* A^x *for the parts of the spectrum inside* $\mathscr{D}$. *Then the triple* $\{(C|\operatorname{Im} P, A|\operatorname{Im} P), (A^x|\operatorname{Im} P^x, P^x B), P^x|\operatorname{Im} P\}$ *is a c.s.s.d. for* W *on* $\mathscr{D}$.

Proof. By definition $(C|\operatorname{Im} P, A|\operatorname{Im} P)$ is a right pole pair for W on $\mathscr{D}$, and we have already observed that W^{-1} has the realization

$$W^{-1}(\lambda) = I_Y - C(\lambda I_X - A^x)^{-1} B$$

and thus $(A^\times|\operatorname{Im} P^\times, P^\times B)$ is a left zero pair for W on $\mathscr{D}$. It remains to characterize the space $M = WH_Y^2$ more explicitly in order to compute the associated pole-zero coupling operator $\hat{T}$.

Thus suppose $f(\lambda) = \sum_{n=1}^\infty f_n \lambda^{n-1} \in H_Y^2$ and $g(\lambda) = W(\lambda) f(\lambda)$. Then $g(\lambda) = \sum_{n=1}^\infty g_n^- \lambda^{-n} + \sum_{n=1}^\infty g_n^+ \lambda^{n-1}$, where g_n^- for $n > 0$ is given by (1.8). Furthermore, for $j > 0$

$$\begin{aligned} g_j^+ &= \sum_{k=1}^\infty T_{j-k} f_k = \sum_{k=1}^{j-1} T_{j-k} f_k + T_0 f_j + \sum_{k=j+1}^\infty T_{j-k} f_k \\ &= -\sum_{k=1}^{j-1} CA^{k-j-1}(I-P)Bf_k + \left\{ I_Y - C\left[A(I-P)\right]^{-1} B \right\} f_j \\ &\quad + \sum_{k=j+1}^\infty CA^{k-j-1}PBf_k \\ &= -\sum_{k=1}^{j} CA^{k-j-1}(I-P)Bf_k + f_j + \sum_{k=j+1}^\infty CA^{k-j-1}PBf_k. \end{aligned}$$

Hence

$$\begin{aligned} &\operatorname{row}\left(A^{\times^{j-1}} P^\times B\right)_{j=1}^\infty \cdot \operatorname{col}\left(g_j^+\right)_{j=1}^\infty \\ &\quad = -\sum_{j=1}^\infty \sum_{k=1}^{j} A^{\times^{j-1}} P^\times BCA^{k-j-1}(I-P)Bf_k \\ &\qquad + \sum_{j=1}^\infty A^{\times^{j-1}} P^\times Bf_j + \sum_{j=1}^\infty \sum_{k=j+1}^\infty A^{\times^{j-1}} P^\times BCA^{k-j-1}PBf_k. \end{aligned}$$

Use $BC = A - A^\times$ and interchange the order of summation to obtain that this equals

$$\begin{aligned} &\sum_{k=1}^\infty \sum_{j=k}^\infty \left(P^\times A^{\times^j} A^{k-j-1}(I-P)B - P^\times A^{\times^{j-1}} A^{k-j}(I-P)B \right) f_k \\ &\quad + \sum_{j=1}^\infty A^{\times^{j-1}} P^\times Bf_j + \sum_{k=2}^\infty \sum_{j=1}^{k-1} \left(P^\times A^{\times^{j-1}} A^{k-j} PB - P^\times A^{\times^j} A^{k-j-1} PB \right) f_k \\ &\quad = -\sum_{k=1}^\infty P^\times A^{\times^{k-1}} (I-P)Bf_k + \sum_{j=1}^\infty A^{\times^{j-1}} P^\times Bf_j \\ &\qquad + \sum_{k=2}^\infty \left(P^\times A^{k-1} PB - P^\times A^{\times^{k-1}} PB \right) f_k \\ &\quad = P^\times \left(\sum_{k=1}^\infty PA^{k-1} PBf_k \right). \end{aligned}$$

Now this should equal

$$\operatorname{row}(A^{\times^{j-1}}P^{\times}B)_{j=1}^{\infty}\cdot T\cdot \operatorname{col}(CA^{j-1}P)_{j=1}^{\infty}\cdot\left(\sum_{k=1}^{\infty}A^{k-1}PBf_k\right)$$

$$=\hat{T}\left(\sum_{k=1}^{\infty}A^{k-1}PBf_k\right).$$

It follows that $\hat{T}=P^{\times}|\operatorname{Im}P$ as claimed. ■

REMARK 1.3. We can now identify the coupling operator $\hat{T}$ associated with a given pole pair (C_+, A_{p+}) and zero pair (A_{z+}, B_+) on $\mathscr{D}$ for a given function $W(\lambda)$. Assume $C_+ = C|\operatorname{Im}P_p$ and $A_{p+}=A_p|\operatorname{Im}P_p$ for a pole pair (C, A_p) for W, where P_p is the Riesz projection of A_p for the unit disk $\mathscr{D}$, and similarly, $A_{z+}=A_z|\operatorname{Im}P_z$ and $B_+ = P_zB$ for a zero pair (A_z, B) for W, where P_z is the Riesz projection for A_z. Thus by definition

$$W(\lambda)=I_Y+C(\lambda I-A_p)^{-1}\tilde{B}$$

and

$$W^{-1}(\lambda)=I_Y-\tilde{C}(\lambda I-A_z)^{-1}B.$$

If we compute W^{-1} directly from the above expression for $W(\lambda)$, we get

$$W^{-1}(\lambda)=I_Y-C(\lambda I-A_p+\tilde{B}C)^{-1}\tilde{B}.$$

By the state space isomorphism theorem for minimal realizations (see e.g. [6]), there must be an invertible linear transformation $S\colon X_p\to X_z$ such that

$$C=\tilde{C}S,$$

$$A_p-\tilde{B}C=S^{-1}A_zS,$$

$$B=S\tilde{B}.$$

If $P^{\times}$ is the Riesz projection of $A_p-\tilde{B}C$ for $\mathscr{D}$, we have from the above that $P^{\times}=S^{-1}P_zS$. From Theorem 1.3 we see that $\{(C_+, A_{p+}), ((A_p-\tilde{B}C)|\operatorname{Im}P^{\times}, P^{\times}B), P^{\times}|\operatorname{Im}P_p\colon \operatorname{Im}P_p\to\operatorname{Im}P^{\times}\}$ is a c.s.s.d. for W on $\mathscr{D}$. Now we use Remark 1.2 to perceive the effect of changing the zero pair

$((A_p - \tilde{B}C)|\operatorname{Im} P^x, P^x\tilde{B})$ to (A_{z+}, B_+). The c.s.s.d. is seen to be $\{(C_+, A_{p+}), (A_{z+}, B_+), P_z S|\operatorname{Im} P_p\}$.

All the above analysis can be done in a similar way for $M^x := WH_Y^{2\perp}$. We define spaces $M_z^x := M^x \cap H_Y^{2\perp}$ and $M_p^x = P_{H_Y^2} M^x$. We say that (C_-, A_{p-}) (where $C_- : X_{p-} \to Y$, $A_{p-} : X_{p-} \to X_{p-}$) is a right pole pair for W on $\mathscr{D}_e := \{\lambda \mid |\lambda| > 1\}$ if $X_{p-} = \operatorname{Im}(I - P_p)$, $C_- = C|\operatorname{Im}(I - P_p)$, and $A_{p-} = A_p|\operatorname{Im}(I - P_p)$ for a right pole pair (C, A_p) for W, where P_p is the Riesz projection for A_p for $\mathscr{D}$. Similarly, (A_{z-}, B_-) (where $A_{z-} : X_{z-} \to X_{z-}$, $B_- : Y \to X_{z-}$) is a left zero pair for W on $\mathscr{D}_e$ if $X_{z-} = \operatorname{Im}(I - P_z)$, $A_{z-} = A|\operatorname{Im}(I - P_z)$, and $B_- = (I - P_z)B$ for a left zero pair (A_z, B) for W. The following analogue of Theorem 1.1 holds.

THEOREM 1.4. *Suppose (C_-, A_{p-}) is a right pole pair for W on $\mathscr{D}_e$ and (A_{z-}, B_-) is a left zero pair for W on $\mathscr{D}_e$. Then if $M_p^x = P_{H_Y^2}(WH_Y^{2\perp})$ and $M_z^x = WH_Y^{2\perp} \cap H_Y^{2\perp}$, we have*

$$\mathscr{F}_+ M_p^x = \operatorname{Im} \operatorname{col}\left(C_- A_{p-}^{-j}\right)_{j=1}^{\infty} \tag{1.12}$$

and

$$\mathscr{F}_- M_z^x = \operatorname{Ker} \operatorname{row}\left(A_{z-}^{-j} B_-\right)_{j=1}^{\infty}. \tag{1.13}$$

The space $M^x := \mathrm{WH}_Y^{2\perp}$ then has the form

$$\mathscr{F} M^x = \left\{(g + T^x f) \oplus f : f \in \mathscr{F}_+ M_p^x,\ g \in \mathscr{F}_- M_z^x\right\} \tag{1.14}$$

for a uniquely determined $T^x : \mathscr{F}_+ M_p^x \to l_Y^2 \ominus \mathscr{F}_- M_z^x$. This operator T^x is in turn uniquely determined by the operator $\hat{T}^x : X_{p-} \to X_{z-}$ given by

$$\hat{T}^x = \operatorname{row}\left(A_{z-}^{-j} B_-\right)_{j=1}^{\infty} \cdot T^x \cdot \operatorname{col}\left(C_- A_{p-}^{-j}\right)_{j=1}^{\infty}. \tag{1.15}$$

We say that the triple $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^x\}$ associated with W in this way is a *canonical set of spectral data* (c.s.s.d.) for W on $\mathscr{D}_e$. The analogue of Theorem 1.2 for $\mathscr{D}_e$ is the following.

THEOREM 1.5. *Suppose $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^x\}$ is a c.s.s.d. for W on $\mathscr{D}_e$. Then $M^x = WH_Y^{2\perp}$ is given by $\mathscr{F} M^x = \{(g + T^x f) \oplus f : f \in$*

$\mathscr{F}_+ M_p^\times$, $g \in \mathscr{F}_- M_z^\times\}$, *where*

$$\mathscr{F}_+ M_p^\times = \operatorname{Im} \operatorname{col}\left(C_- A_{p-}^{-j}\right)_{j=1}^{\infty},$$

$$\mathscr{F}_- M_z^\times = \operatorname{Ker} \operatorname{row}\left(A_{z-}^{-j} B_-\right)_{j=1}^{\infty},$$

and $T^\times : \mathscr{F}_+ M_p^\times \to l_Y^2 \ominus \mathscr{F}_- M_z^\times$ *is the unique solution of*

$$\hat{T}^\times = \operatorname{row}\left(A_{z-}^{-j} B_-\right)_{j=1}^{\infty} \cdot T^\times \cdot \operatorname{col}\left(C_- A_{p-}^{-j}\right)_{j=1}^{\infty}.$$

If $S_1 : X'_{p-} \to X_{p-}$ and $S_2 : X_{z-} \to X'_{z-}$ are linear bijections and $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^\times\}$ is a c.s.s.d. for W on $\mathscr{D}_e$ where $A_{p-} : X_{p-} \to X_{p-}$ and $A_{z-} : X_{z-} \to X_{z-}$, then $\{(C_- S_1, S_1^{-1} A_{p-} S_1), (S_2 A_{z-} S_2^{-1}, S_2 B_-), S_2 \hat{T}^\times S_1\}$ is another c.s.s.d. for W on $\mathscr{D}_e$, and any two c.s.s.d.'s for W on $\mathscr{D}_e$ are related in this way.

Finally the analogue of Theorem 1.3 is as follows.

THEOREM 1.6. *Suppose $W(\lambda) = I_Y + C(\lambda I_X - A)^{-1} B$ is a minimal realization for the rational matrix function W which is analytic and invertible on the unit circle. Let $A^\times := A - BC$, P, and $P^\times$ be as in Theorem 1.3. Then the triple $\{(C \mid \operatorname{Im}(I - P), A \mid \operatorname{Im}(I - P)), (A^\times \mid \operatorname{Im}(I - P^\times), (I - P^\times)B), (P^\times - I) \mid \operatorname{Im}(I - P)\}$ is a c.s.s.d. for W on $\mathscr{D}_e$.*

In general if (C_-, A_{p-}) is a pole pair on $\mathscr{D}_e$ of the form $C_- = C \mid \operatorname{Im}(I - P_p)$, $A_{p-} = A_p \mid \operatorname{Im}(I - P_p)$ for a pole pair (C, A_p) for W, and (A_{z-}, B_-) is a zero pair on $\mathscr{D}_e$ of the form $A_{z-} = A_z \mid \operatorname{Im}(I - P_z)$ and $B_- = (I - P_z)B$ for a zero pair (A_z, B) for W, then as in Remark 1.3 there is a similarity $S : X_p \to X_z$ such that

$$C = \tilde{C} S,$$

$$A_p - \tilde{B} C = S^{-1} A_z S,$$

$$B = S \tilde{B}.$$

Then one easily sees, as in Remark 1.3, that a c.s.s.d. on $\mathscr{D}_e$ for W is given by $\{(C_-, A_{p-}), (A_{z-}, B_-), (P_z - I) S \mid \operatorname{Im}(I - P_p)\}$.

2. INVERSE SPECTRAL PROBLEM ON $\mathscr{D}$ OR $\mathscr{D}_e$

In this section we consider the following inverse spectral problem. We are given bounded linear operators $C_+ : X_{p+} \to Y$, $A_{p+} : X_{p+} \to X_{p+}$, $A_{z+} : X_{z+} \to X_{z+}$, $B_+ : Y \to X_{z+}$, and $\hat{T}: X_{p+} \to X_{z+}$ such that

$$\sigma(A_{p+}) \subset \mathscr{D} := \{z \mid |z| < 1\}, \qquad \sigma(A_{z+}) \subset \mathscr{D},$$

$$\operatorname{Ker} \operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty} = (0), \tag{2.1}$$

$$\operatorname{Im} \operatorname{row}\left(A_{z+}^{j-1} B_+\right)_{j=1}^{\infty} = X_{z+}. \tag{2.2}$$

Here all spaces are assumed to be finite dimensional. The question is whether there exists a rational matrix function W analytic and invertible on $\{|z| = 1\}$ with values acting on Y such that the triple $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. for W on $\mathscr{D}$. The result is as follows.

THEOREM 2.1. *Let $C_+ : X_{p+} \to Y$, $A_{p+} : X_{p+} \to X_{p+}$, $A_{z+} : X_{z+} \to X_{z+}$, $B_+ : Y \to X_{z+}$, and $\hat{T}: X_{p+} \to X_{z+}$ be given such that* (2.1) *and* (2.2) *are satisfied and $\sigma(A_{p+}) \subset \mathscr{D}$, $\sigma(A_{z+}) \subset \mathscr{D}$. Then $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. on $\mathscr{D}$ for some rational matrix function W if and only if $\hat{T}$ satisfies the Lyapunov equation*

$$A_{z+}\hat{T} - \hat{T}A_{p+} = -B_+C_+. \tag{2.3}$$

If W' is another rational matrix function with the same c.s.s.d. on $\mathscr{D}$, then $W'(\lambda) = W(\lambda)F(\lambda)$, where $F(\lambda)$ is a rational matrix function which is analytic and invertible on the closed unit disk.

Proof. If W and W' are two rational matrix functions with the same c.s.s.d. on $\mathscr{D}$, then by Theorem 1.2 $WH_Y^2 = W'H_Y^2$, or $W^{-1}W'H_Y^2 = H_Y^2$. This forces $F := W^{-1}W'$ to be analytic and invertible on the closed unit disk. Thus the uniqueness statement follows easily.

Now suppose that C_+, A_{p+}, A_{z+}, B_+, and $\hat{T}$ are given as above. We define subspaces $M_p \subset H_Y^{2\perp}$ and $M_z \subset H_Y^2$ by

$$\mathscr{F}_- M_p = \operatorname{Im} \operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty}$$

and

$$\mathscr{F}_+ M_z = \operatorname{Ker} \operatorname{row}\left(A_{z+}^{j-1} B_+\right)_{j=1}^{\infty}.$$

We then define the operator $T\colon \mathscr{F}_- M_p \to l_Y^2 \ominus \mathscr{F}_+ M_z$ by the equation (1.11):

$$\hat{T} = \operatorname{row}\left(A_{z+}^{n-1} B_+\right)_{n=1}^{\infty} \cdot T \cdot \operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty}, \tag{2.4}$$

and define the subspace $M \subset L_Y^2$ by (1.10):

$$\mathscr{F}M = \left\{ f \oplus (Tf + g) : f \in \mathscr{F}_- M_p,\ g \in \mathscr{F}_+ M_z \right\}. \tag{2.5}$$

By Theorem 1.2, $M = WH_Y^2$ if W is a solution of the inverse spectral problem. In particular, a necessary condition for such a W to exist is that M be invariant under the shift operator $M_\lambda\colon f(\lambda) \to \lambda f(\lambda)$ on L_Y^2. The following lemma characterizes when this is the case.

LEMMA 2.2. *Let the spectral data set* $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ *be given, and define the subspace* $M \subset L_Y^2$ *from this data set as above. Then* M *is invariant if and only if Equation* (2.3) *holds.*

Proof. If $S = \mathscr{F} M_\lambda \mathscr{F}^{-1}$ is the shift operator on L_Y^2 represented on $l_Y^2 \oplus l_Y^2$ via the identification map $\mathscr{F}$, then

$$S = \begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix},$$

where S_1 is the backward unilateral shift on l_Y^2, S_2 is the projection to the first coordinate, and S_3 is the forward unilateral shift. We write $\mathscr{F}M$ in block matrix form as

$$\mathscr{F}M = \operatorname{Im}\begin{pmatrix} I & 0 \\ T & I \end{pmatrix}\Bigg|_{\mathscr{F}_- M_p \oplus \mathscr{F}_+ M_z},$$

where T is determined from $\hat{T}$ by (2.1). Invariance of $\mathscr{F}M$ under S means that for any $f \in \mathscr{F}_- M_p$ and $g \in \mathscr{F}_+ M_z$ there exists $f' \in \mathscr{F}_- M_p$ and $g' \in \mathscr{F}_+ M_z$ such that

$$\begin{pmatrix} S_1 & 0 \\ S_2 & S_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ T & I \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix}\begin{pmatrix} f' \\ g' \end{pmatrix},$$

i.e., $f' = S_1 f$ and

$$(S_3 T - T S_1) f + S_2 f \in \mathscr{F}_+ M_z.$$

From the definition of M_z one sees that this is equivalent to

$$\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}(S_3T - TS_1)f = -\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}S_2 f.$$

Since $f \in \mathscr{F}_- M_p$, we can write $f = \operatorname{col}(C_+ A_{p+}^{j-1})_{j=1}^{\infty} w$ with $w \in X_p$. Hence M is invariant if and only if for every $w \in X_{p+}$ we have

$$\begin{aligned}&\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}S_3T\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}w\\ &\quad-\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}TS_1\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}w\\ &\quad+\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}S_2\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}w=0.\end{aligned}\tag{2.6}$$

By the definition of S_2 one easily sees that

$$\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}S_2\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}=B_+C_+.$$

Check also that

$$\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}S_3=A_{z+}\operatorname{row}\left(A_{z+}^{j-1}B_+\right)_{j=1}^{\infty}$$

and

$$S_1\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}=\operatorname{col}\left(C_+A_{p+}^{j-1}\right)_{j=1}^{\infty}A_{p+}.$$

Combine these relations with the defining relation (2.1) for T to see that (2.6) is equivalent to (2.3). ■

Continuation of the proof of Theorem 2.1. Let us now assume that Equation (2.3) holds, so M is invariant under M_λ. Since $P_{H_Y^{2\perp}}M = M_p$ and $H_Y^2 \ominus (M \cap H_Y^2) = H_Y^2 \ominus M_z$ are both finite dimensional, it is not difficult to see that in fact M is *full range simply invariant* [i.e., $\bigcup\{M_\lambda^{-n}M \mid n = 1,2,\dots\}$ is dense in L_Y^2 and $\bigcap_{n=0}^{\infty} M_\lambda^n M = (0)$]. Then by the Beurling-Lax theorem (see e.g. [11]), there is a matrix function $W(\lambda)$ on the unit circle such that $M = WH_n^2$; in fact there is a W with the additional property that $W(\lambda)$ is unitary for $|\lambda| = 1$. By the finite-dimensionality of M_p and $H_Y^2 \ominus M_z$, any such unitary valued W is rational (see e.g. [2]). It remains to show that W is the solution of the inverse spectral problem. We remark that the unitary valued

W with $WH^2_Y = M$ is analytic and invertible at ∞ if and only if both A_{p+} and A_{z+} are invertible. If this is not the case, we can use $W' = WF$ in place of W, where F is any rational matrix function analytic and invertible on the closed unit disk with a pole and/or zero at ∞ designed to cancel out any zero and/or pole of W at ∞. By then multiplying W' on the right by an invertible constant matrix we may assume that $W'(\infty) = I_Y$. Let us assume that all this has been done. Then W has a minimal realization $W(\lambda) = I_Y + C'(\lambda I_{X'} - A')^{-1}B'$ for some $C': X' \to Y$, $A': X' \to X'$, $B': Y \to X'$. Let (C'_+, A'_{p+}) be the right pole pair $(C'|\operatorname{Im} P', A'|\operatorname{Im} P')$ for W on $\mathscr{D}$, and (A'_{z+}, B'_+) be the left zero pair $(A'^{\times}|\operatorname{Im} P'^{\times}, P'^{\times}B')$ for W on $\mathscr{D}$, where $A'^{\times} := A' - B'C'$ and P' and $P'^{\times}$ are the appropriate Riesz projections.

By the direct analysis (Theorem 1.1), we know that

$$\mathscr{F}_- M_p = \operatorname{Im} \operatorname{col}\left(C_+ A_{p+}^{j-1}\right)_{j=1}^{\infty} = \operatorname{Im} \operatorname{col}\left(C'_+ A'^{j-1}_{p+}\right)_{j=1}^{\infty}.$$

Since each of $\operatorname{col}(C_+ A_{p+}^{j-1})_{j=1}^{\infty}$ and $\operatorname{col}(C'_+ A'^{j-1}_{p+})_{j=1}^{\infty}$ is injective, there must be a bijective linear mapping $S: X_{p+} \to \operatorname{Im} P'$ such that $\operatorname{col}(C_+ A_{p+}^{j-1})_{j=1}^{\infty} = \operatorname{col}(C'_+ A'^{j-1}_{p+})\, S$, i.e., $C_+ A_{p+}^{j-1} = C'_+ A'^{j-1}_{p+} S$ for $j = 1, 2, \ldots$. But then $C_+(\lambda I_{X_{p+}} - A_{p+})^{-1} = C'_+(\lambda I_{\operatorname{Im} P'} - A'_{p+})^{-1} S$. From this one easily sees that (C_+, A_{p+}) is indeed a right pole pair for W on $\mathscr{D}$.

To see that (A_{z+}, B_+) is a left zero pair for W on $\mathscr{D}$, note that by the direct analysis (Theorem 1.1)

$$\mathscr{F}_+ M_z = \operatorname{Ker} \operatorname{row}\left(A_{z+}^{j-1} B_+\right)_{j=1}^{\infty} = \operatorname{Ker} \operatorname{row}\left(A'^{j-1}_{z+} B'_+\right)_{j=1}^{\infty}.$$

Since both $\operatorname{row}(A_{z+}^{j-1}B_+)_{j=1}^{\infty}$ and $\operatorname{row}(A'^{j-1}_{z+}B'_+)_{j=1}^{\infty}$ are surjective, there must be a linear bijection $S': \operatorname{Im} P'^{\times} \to X_{z+}$ such that

$$\operatorname{row}\left(A_{z+}^{j-1} B_+\right)_{j=1}^{\infty} = S' \operatorname{row}\left(A'^{j-1}_{z+} B'_+\right)_{j=1}^{\infty},$$

i.e., $A_{z+}^{j-1}B_+ = S'A'^{j-1}_{z+}B'_+$ for $j = 1, 2, \ldots$. But then $(\lambda I_{X_{z+}} - A_{z+})^{-1}B_+ = S'(\lambda I - A'_{z+})^{-1}B'_+$. This implies that (A_{z+}, B_+) is a left zero pair for W on $\mathscr{D}$.

Finally, $\hat{T}$ is the coupling operator for W on $\mathscr{D}$ going with this pole and zero pair, since $M = WH^2_Y$ and M is defined by (2.5) and (2.4). ∎

An analogous characterization of c.s.s.d.'s on $\mathscr{D}_e$ holds. The proof is completely analogous to that above for $\mathscr{D}$, so it will be omitted.

THEOREM 2.3. *Let* $C_-: X_{p-} \to Y$, $A_{p-}: X_{p-} \to X_{p-}$, $A_{z-}: X_{z-} \to X_{z-}$, $B_-: Y \to X_{z-}$, *and* $\hat{T}^x: X_{p-} \to X_{z-}$ *be given such that*

$$\operatorname{Im} \operatorname{row}\left(A_{z-}^j B_-\right)_{j=0}^{\infty} = X_{z-} \tag{2.7a}$$

and

$$\operatorname{Ker} \operatorname{col}\left(C_- A_{z-}^j\right)_{j=0}^{\infty} = (0) \tag{2.7b}$$

and $\sigma(A_{p-}) \subset \mathscr{D}_e$, $\sigma(A_{z-}) \subset \mathscr{D}_e$. *Then* $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^x\}$ *is a c.s.s.d. on* $\mathscr{D}_e$ *for some rational matrix function* W *if and only if*

$$A_{z-}\hat{T}^x - \hat{T}^x A_{p-} = B_- C_-. \tag{2.8}$$

It is also possible to formulate a weaker inverse spectral problem on $\mathscr{D}$; namely, we are given operators C_+, A_{p+}, A_{z+}, and B_+ and are asked if there is a rational matrix function W such that (C_+, A_{p+}) is a right pole pair on $\mathscr{D}$ for W and (A_{z+}, B_+) is a left zero pair on $\mathscr{D}$ for W; this is more in the spirit of the problem studied in [9]. If W is a solution of this problem, then a unique coupling operator $\hat{T}$ on $\mathscr{D}$ exists for W corresponding to (C_+, A_{p+}) and (A_{z+}, B_+), given by (1.11) and (1.10) with $M = WH_Y^2$. Thus the question can be reformulated as follows: given (C_+, A_{p+}) and (A_{z+}, B_+), when does there exist an operator $\hat{T}: X_{p+} \to X_{z+}$ such that $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. on $\mathscr{D}$ for some rational matrix function W. At the level of invariant subspaces, given a subspace $M_p \subset H_Y^{2\perp}$ and a subspace $M_z \subset H_Y^2$, the problem is to construct an invariant subspace M such that $M_p = P_{H_Y^{2\perp}} M$ and $M_z = H_Y^2 \cap M$. The analogous problem can be formulated for $\mathscr{D}_e$. The following result follows immediately from Theorems 2.1 and 2.3 combined with the above remarks.

COROLLARY 2.4.

(a) *Suppose* $C_+: X_{p+} \to Y$, $A_{p+}: X_{p+} \to X_{p+}$, $A_{z+}: X_{z+} \to X_{z+}$, *and* $B: Y \to X_{z+}$ *are given, where* (2.1) *and* (2.2) *are satisfied, and* $\sigma(A_{z+}) \subset \mathscr{D}$, $\sigma(A_{p+}) \subset \mathscr{D}$. *Then there exists a rational matrix function* W *such that* (C_+, A_{p+}) *is a right pole pair on* $\mathscr{D}$ *for* W *and* (A_{z+}, B_+) *is a left zero pair on* $\mathscr{D}$ *for* W *if and only if there exists a solution* $\hat{T}: X_{p+} \to X_{z+}$ *of the Lyapunov equation* (2.3). *If two solutions* W *and* W' *are considered equivalent if* $W' = WF$, *where* F *is analytic and invertible on the closed unit disk, then equivalence classes of solutions are in one-to-one correspondence with solutions* $\hat{T}$ *of* (2.3).

(b) *Suppose* $C_-: X_{p-} \to Y$, $A_{p-}: X_{p-} \to X_{p-}$, $A_{z-}: X_{z-} \to X_{z-}$ *and* $B_-: Y \to X_{z-}$ *are given, where* (2.6) *and* (2.7) *are satisfied and* $\sigma(A_{p-}) \subset \mathscr{D}_e$, $\sigma(A_{z-}) \subset \mathscr{D}_e$. *Then there exists a rational matrix function* W *such that* (C_-, A_{p-}) *is a right pole pair on* $\mathscr{D}_e$ *for* W *and* (A_{z-}, B_-) *is a left zero pair on* $\mathscr{D}_e$ *for* W *if and only if there exists a solution* $\hat{T}^\times: X_{p-} \to X_{z-}$ *of the Lyapunov equation* (2.8). *If two solutions* W *and* W' *are considered equivalent if* $W' = WF$ *where* F *if analytic and invertible on* $\mathscr{D}_e$ *(including* ∞*), then equivalence classes of solutions are in one-to-one correspondence with solutions* $\hat{T}^\times$ *of* (2.8).

3. INVERSE SPECTRAL PROBLEM ON $\mathbb{C}$

Combining the two inverse spectral problems of the previous section, we arrive at the following problem. Given are C_-, A_{p+}, A_{z+}, B_+, $\hat{T}$ along with C_-, A_{p-}, A_{z-}, B_-, $\hat{T}^\times$ acting on appropriate spaces. The question is whether there exists a rational matrix function W with $W(\infty) = I_Y$ such that $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. on $\mathscr{D}$ for W and $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^\times\}$ is a c.s.s.d. on $\mathscr{D}_e$ for the same W. Theorems 2.1 and 2.3 give a pair of necessary conditions [(2.3) and (2.8)]. It turns out that these alone are not sufficient. The result is as follows.

THEOREM 3.1. *Suppose* C_+, A_{p+}, A_{z+}, B_+, $\hat{T}$ *along with* C_-, A_{p-}, A_{z-}, B_-, $\hat{T}^\times$ *acting on appropriate spaces are given such that* $\sigma(A_{p+}) \cup \sigma(A_{z+}) \subset \mathscr{D}$, $\sigma(A_{p-}) \cap \sigma(A_{z-}) \subset \mathscr{D}_e$, *and the observability and controllability conditions* (2.1), (2.2), *and* (2.7) *hold. Then there is a rational matrix function* W *with* $W(\infty) = I_Y$ *such that* $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ *is a c.s.s.d. on* $\mathscr{D}$ *for* W *and* $\{(C_-, A_{p-}), (A_{z-}, B_-), \hat{T}^\times\}$ *is a c.s.s.d. on* $\mathscr{D}_e$ *for the same* W *if and only if*

(i) $\hat{T}$ *satisfies* (2.3),
(ii) $\hat{T}^\times$ *satisfies* (2.8), *and*
(iii) *the operator* $\hat{\mathscr{T}}: X_{p-} \dotplus X_{p+} \to X_{z-} \dotplus X_{z+}$ *defined by*

$$\hat{\mathscr{T}} = \begin{pmatrix} -\hat{T}^\times & -\sum_{j=1}^{\infty} A_{z-}^{-j} B_- C_+ A_{p+}^{j-1} \\ \sum_{j=1}^{\infty} A_{z+}^{j-1} B_+ C_- A_{p-}^{-j} & \hat{T} \end{pmatrix} \tag{3.1}$$

is invertible. Moreover, in this case the solution W *with* $W(\infty) = I_Y$ *is* *unique and is given by*

$$W(\lambda)=I_Y+C\left(\lambda I_{X_p}-A_p\right)^{-1}\hat{\mathscr{T}}^{-1}B \tag{3.2}$$

with

$$W(\lambda)^{-1}=I-C\hat{\mathscr{T}}^{-1}(\lambda I_{X_z}-A_z)^{-1}B. \tag{3.3}$$

Here $X_p:=X_{p-}\dot{+}X_{p+}$, $X_z:=X_{z-}\dot{+}X_{z+}$, $C=(C_- \ \ C_+)$,

$$A_z=\begin{pmatrix} A_{z-} & 0 \\ 0 & A_{z+}\end{pmatrix},\qquad A_p=\begin{pmatrix} A_{p-} & 0 \\ 0 & A_{p+}\end{pmatrix},\quad \textit{and}\quad B=\begin{pmatrix} B_- \\ B_+\end{pmatrix}.$$

Proof. Assume $\hat{T}$ satisfies (2.3), and construct the subspace $M\subset L^2_Y$ as in Theorem 2.1; then M is a full range simply invariant subspace of L^2_Y of rational type. Assume $\hat{T}^x$ satisfies (2.8), and construct the subspace $M^x\subset L^2_Y$ as in Theorem 2.3 [using (1.12), (1.13), (1.14), and (1.15)]; then M^x is full range and simply invariant for the shift $(M_\lambda)^{-1}$, of rational type and regular at ∞. The problem is to produce a rational matrix function W with $W(\infty)=I_Y$ such that $M^x=WH^{2\perp}_Y$ and $M=WH^2_Y$. By the main result in [2], this is possible if and only if the matching condition

$$L^2_Y=M^x\dot{+}M$$

holds, or equivalently, if and only if

$$\mathscr{F}M+\mathscr{F}M^x=l^2_Y\oplus l^2_Y \tag{3.4}$$

and

$$\mathscr{F}M\cap\mathscr{F}M^x=(0). \tag{3.5}$$

To analyze these conditions, write elements $f\oplus g$ of $l^2_Y\oplus l^2_Y$ in column form $\begin{pmatrix} f \\ g\end{pmatrix}$. Then we can write

$$\mathscr{F}M=\begin{pmatrix} I & 0 \\ T & I\end{pmatrix}\begin{pmatrix}\mathscr{F}_-M_p \\ \mathscr{F}_+M_z\end{pmatrix}$$

and

$$\mathscr{F}M^{x} = \begin{pmatrix} I & T^{x} \\ 0 & I \end{pmatrix} \begin{pmatrix} \mathscr{F}_{-} M_{z}^{x} \\ \mathscr{F}_{+} M_{p}^{x} \end{pmatrix}.$$

Let $\mathscr{G}$ denote the subspace $\mathscr{G}_{-} \oplus \mathscr{G}_{+}$ of $l_Y^2 \oplus l_Y^2$, where $\mathscr{G}_{-} = l_Y^2 \ominus \mathscr{F}_{-} M_z^x$ and $\mathscr{G}_{+} = l_Y^2 \ominus \mathscr{F}_{+} M_z$. Then the orthogonal projection of $l_Y^2 \oplus l_Y^2$ onto $\mathscr{G}$ can be written as

$$P = \begin{pmatrix} P_{-} & 0 \\ 0 & P_{+} \end{pmatrix},$$

where $P_{\pm} : l_Y^2 \to \mathscr{G}_{\pm}$ are the orthogonal projections. Introduce the operator $\mathscr{T} : \mathscr{F}_{+} M_p^x \oplus \mathscr{F}_{-} M_p \to \mathscr{G}_{-} \oplus \mathscr{G}_{+}$ by

$$\mathscr{T} = \begin{pmatrix} P_{-} T^{x} & P_{-} \\ P_{+} & P_{+} T \end{pmatrix}.$$

Note that if $\hat{\mathscr{T}}$ is as in (3.1), then

$$\hat{\mathscr{T}} = \begin{pmatrix} -\operatorname{row}\left(A_{z-}^{-j} B_{-}\right)_{j=1}^{\infty} & 0 \\ 0 & \operatorname{row}\left(A_{z+}^{j-1} B_{+}\right)_{j=1}^{\infty} \end{pmatrix}$$

$$\cdot \mathscr{T} \cdot \begin{pmatrix} \operatorname{col}\left(C_{-} A_{p-}^{-j}\right)_{j=1}^{\infty} & 0 \\ 0 & \operatorname{col}\left(C_{+} A_{p+}^{j-1}\right)_{j=1}^{\infty} \end{pmatrix}. \tag{3.6}$$

Also the first factor is a bijection from $\mathscr{G}_{-} \oplus \mathscr{G}_{+}$ onto $X_{z-} \dotplus X_{z+}$ and the last factor is a bijection from $X_{p-} \dotplus X_{p+}$ onto $\mathscr{F}_{-} M_p^x \oplus \mathscr{F}_{+} M_p$. We conclude that $\hat{\mathscr{T}}$ is invertible if and only if $\mathscr{T}$ is. Thus we need only show that (3.4) and (3.5) holding is equivalent to the invertibility of $\mathscr{T}$.

We consider (3.4) first. From the form of $\mathscr{F}M$ and $\mathscr{F}M^x$ we see that (3.4) holds if and only if $P\mathscr{F}M^x + P\mathscr{F}M = \mathscr{G}$. Note that $P\mathscr{F}M^x$ collapses to

$$\begin{pmatrix} P_{-} T^{x} \\ P_{+} \end{pmatrix} \mathscr{F}_{+} M_{p}^{x}$$

and that $P\mathscr{F}M$ collapses to

$$\begin{pmatrix} P_- \\ P_+T \end{pmatrix} \mathscr{F}_- M_p .$$

From this it is clear that (3.4) is equivalent to the surjectivity of $\mathscr{T}$.

Next suppose that $f_+ \dot{+} f_- \in \operatorname{Ker} \mathscr{T}$. Thus $f_+ \in \mathscr{F}_+ M_p^x$, $f_- \in \mathscr{F}_- M_p$, and

$$\begin{aligned} P_-(T^x f_+ + f_-) &= 0, \\ P_+(f_+ + Tf_-) &= 0. \end{aligned} \tag{3.7}$$

Thus $g_- := T^x f_+ + f_- \in \mathscr{F}_- M_z^x$ and $g_+ := f_+ + Tf_- \in \mathscr{F}_+ M_z$. Thus $f_- \oplus (Tf_- - g_+) \in \mathscr{F}M$. But from (3.7) we get $f_- = g_- - T^x f_+$ and $Tf_- - g_+ = -f_+$, and thus on the other hand that $f_- \oplus (Tf_- - g_+) = (g_- - T^x F_+) \oplus (-f_+) \in \mathscr{F}M^x$. Thus if (3.5) holds, we get $f_- = 0$ and $f_+ = 0$ and hence $\operatorname{Ker} \mathscr{T} = (0)$.

Conversely suppose $\operatorname{Ker} \mathscr{T} = (0)$, and suppose $h_- \oplus h_+ \in \mathscr{F}M^x \cap \mathscr{F}M$. Then there exist $g_- \in \mathscr{F}_- M_z^x$, $f_+ \in \mathscr{F}_+ M_p^x$, $f_- \in \mathscr{F}_- M_p$, and $g_+ \in \mathscr{F}_+ M_z$ such that

$$h_- \oplus h_+ = (g_- + T^x f_+) \oplus f_+ = f_- \oplus (Tf_- + g_+).$$

Thus

$$\begin{aligned} g_- + T^x f_+ &= f_-, \\ f_+ &= Tf_- + g_+. \end{aligned}$$

Apply P_- to the first equation and P_+ to the second to get

$$\begin{aligned} P_- T^x f_+ &= P_- f_-, \\ P_+ f_+ &= P_+ Tf_-. \end{aligned}$$

From this we get that $f_+ \dot{+} -f_-$ is in $\operatorname{Ker}\mathscr{T}$. From the hypothesis $\operatorname{Ker}\mathscr{T} = (0)$ we get $f_+ = 0$, $f_- = 0$. From this we see that $h_- \oplus h_+ = 0$, so (3.5) holds.

We have shown that the inverse spectral problem of the theorem has a solution if and only if $\hat{\mathscr{T}}$ is invertible. If W' and W are two solutions, then $F = W'^{-1}W$ would be analytic and invertible on all of $\mathbb{C}$ with $F(\infty) = I_Y$. Thus by Liouville's theorem, $F \equiv I_Y$ and $W' = W$.

It remains to show that W is given by (3.2) and (3.3). Define W by (3.2). The first order of business is to show that W^{-1} is given by (3.3). From (3.2) we get

$$W(\lambda)^{-1} = I_Y - C\left(\lambda I_{X_p} - A_p + \hat{\mathscr{T}}^{-1}BC\right)^{-1}\hat{\mathscr{T}}^{-1}B$$

$$= I_Y - C\hat{\mathscr{T}}^{-1}\left(\lambda I_{X_z} - \hat{\mathscr{T}}A_p\hat{\mathscr{T}}^{-1} + BC\hat{\mathscr{T}}^{-1}\right)^{-1}B.$$

A direct computation shows that $\hat{\mathscr{T}}$ satisfies the Lyapunov equation

$$\hat{\mathscr{T}}A_p - A_z\hat{\mathscr{T}} = BC. \tag{3.8}$$

Thus

$$\hat{\mathscr{T}}A_p\hat{\mathscr{T}}^{-1} - BC\hat{\mathscr{T}}^{-1} = A_z, \tag{3.9}$$

and (3.3) follows. From (3.2) it is clear that (C_+, A_{p+}) and (C_-, A_{p-}) are right pole pairs for W on $\mathscr{D}$ and $\mathscr{D}_e$ respectively, and from (3.3) it is clear that (A_{z+}, B_+) and (A_{z-}, B_-) are left zero pairs for W on $\mathscr{D}$ and $\mathscr{D}_e$ respectively. Moreover, from (3.9) we see that the similarity S linking these zero and pole pairs as in Remark 1.3 is $S = \hat{\mathscr{T}}$. Then by Remark 1.3 we see that the coupling operator for W associated with pole pair (C_+, A_{p+}) and zero pair (A_{z+}, B_+) on $\mathscr{D}$ is given by

$$P_z\hat{\mathscr{T}}|\operatorname{Im} P_p = \begin{pmatrix} 0 & I \end{pmatrix}\hat{\mathscr{T}}|\operatorname{Im}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

By (3.1), this equals the preassigned operator $\hat{T}$. Similarly the coupling operator for W associated with pole pair (C_-, A_{p-}) and zero pair (A_{z-}, B_-) on $\mathscr{D}_e$ by the discussion after Theorem 1.6 is given by

$$(P_z - I)\hat{\mathscr{T}}|\operatorname{Im}(I - P_p) = \begin{pmatrix} 0 & -I \end{pmatrix}\hat{\mathscr{T}}|\operatorname{Im}\begin{pmatrix} I \\ 0 \end{pmatrix}.$$

By (3.1) again, this in turn equals the prescribed $\hat{T}^{\times}$. Thus W defined by (3.2) and (3.3) is a solution of the inverse spectral problem as claimed. ∎

REMARK. Note that the verification that W given by (3.2) solves the inverse spectral problem in Theorem 3.1 did not use the result from [2]. This gives a different independent proof of the nontrivial direction of the main result from [2], at least for the rational case.

We can formulate a weaker inverse spectral problem on $\mathbf{C}$, as was done in Section 2 for the inverse problems on $\mathscr{D}$ and $\mathscr{D}_e$. We are given operators C, A_z, A_p, B and ask when there is a rational matrix function W with $W(\infty)=I_Y$ such that (C, A_p) is a right pole pair for W and (A_z, B) is a left zero pair. This is the problem considered in [9]. We obtain their result as a corollary of Theorem 3.1. Indeed this result suggested the formula for the solution of the inverse spectral problem in Theorem 3.1.

COROLLARY 3.2. *Suppose (C, A_p) is observable and (A_z, B) is controllable, where $C: X_p \to Y$, $A_p: X_p \to X_p$, $A_z: X_z \to X_z$, and $B: Y \to X_z$. Then there is a rational matrix function W with $W(\infty)=I_Y$ such that (C, A_p) is a right pole pair and (A_z, B) is a left zero pair for W if and only if there exists a solution $\hat{\mathscr{T}}$ of the Lyapunov equation* (3.8)

$$\hat{\mathscr{T}}A_p - A_z\hat{\mathscr{T}} = BC$$

which is invertible. Solutions W of this inverse spectral problem are in one-to-one correspondence with invertible solutions $\hat{\mathscr{T}}$ of (3.8) *via Equations* (3.2) *and* (3.3).

Proof. It is no essential loss of generality to assume that A_p and A_z have no spectrum on the unit circle. Then via the Riesz projections we can represent X_p as $X_{p-} \dot{+} X_{p+}$ and X_z as $X_{z-} \dot{+} X_{z+}$, so that C, A_p, A_z, B have the form

$$C = \begin{pmatrix} C_+ & C_- \end{pmatrix},$$

$$A_p = \begin{pmatrix} A_{p-} & 0 \\ 0 & A_{p+} \end{pmatrix},$$

$$A_z = \begin{pmatrix} A_{z-} & 0 \\ 0 & A_{z+} \end{pmatrix},$$

$$B = \begin{pmatrix} B_- \\ B_+ \end{pmatrix},$$

where $\sigma(A_{p-}) \cup \sigma(A_{z-}) \subset \mathscr{D}_e$ and $\sigma(A_{p+}) \cup \sigma(A_{z+}) \subset \mathscr{D}$. Then the problem is to solve the two inverse spectral problems discussed in Corollary 2.4, but simultaneously with a single function W. Thus it is necessary that there exist solutions $\hat{T}$ and $\hat{T}^\times$ of (2.3) and (2.8). By Theorem 3.1, it is necessary and sufficient that, for some such choice of $\hat{T}$ and $\hat{T}^\times$, the operator $\hat{\mathscr{T}}$ defined by (3.1) is invertible, and solutions W are in one-to-one correspondence with such solution pairs $(\hat{T}, \hat{T}^\times)$. It was observed in the proof of Theorem 3.1 that any such $\hat{\mathscr{T}}$ is a solution of (3.8). Conversely, it is not hard to show that any solution $\hat{\mathscr{T}}$ of (3.8) must have the form (3.1), where $\hat{T}^\times$ is a solution of (2.8) and $\hat{T}$ is a solution of (2.3). The corollary follows. ∎

4. APPLICATIONS

4.1. Wiener-Hopf Factorization

Suppose we are given a rational matrix function $W(\lambda)$ analytic and invertible on the unit circle for which we know a minimal realization

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B. \tag{4.1}$$

We would like to know when $W(\lambda)$ admits a canonical Wiener-Hopf factorization with respect to the unit circle, i.e. a factorization of the form

$$W(\lambda) = W_-(\lambda)W_+(\lambda) \tag{4.2}$$

where $W_+(\lambda)$ is a rational function analytic and invertible on the closure $\bar{\mathscr{D}}$ of $\mathscr{D}$, and $W_-(\lambda)$ is a rational function analytic and invertible on $\bar{\mathscr{D}}_e$ (including ∞). For such a factorization to exist, we must have

$$W_- H_Y^2 = W H_Y^2, \qquad W_- H_Y^{2\perp} = H_Y^{2\perp}. \tag{4.3}$$

Conversely, suppose there exists a rational matrix function W_- which satisfies (4.3). Put $W_+(\lambda) := W_-(\lambda)^{-1}W(\lambda)$. It is easily verified that then $W = W_- W_+$ is a canonical Wiener-Hopf factorization. Now (4.3) can be read as an inverse spectral problem on $\mathbb{C}$ for the unknown function W_-. Indeed, the first condition says that a c.s.s.d. on $\mathscr{D}$ for W_- must be equal to a c.s.s.d. on $\mathscr{D}$ for the known function W; from the realization (4.1) we know that such a set of data is given by $\{(C|\operatorname{Im} P, A|\operatorname{Im} P), (A^\times|\operatorname{Im} P^\times, P^\times B), P^\times|\operatorname{Im} P\}$, where P is the Riesz projection for A and $P^\times$ for $A^\times := A - BC$ for spectrum in $\mathscr{D}$. The second condition in (4.3) means that a c.s.s.d. on $\mathscr{D}_e$ for W_- must be the same as c.s.s.d. on $\mathscr{D}_e$ for the constant identity function. It is easy to see that a c.s.s.d. on $\mathcal{D}_e$ for I_Y is the trivial triple $\{(0,0), (0,0), 0\}$ where the state spaces X_{p-} and X_{z-} involved are both the zero space (0). Using Theorem 3.1 we arrive at the following.

THEOREM 4.1. *There exists a canonical Wiener-Hopf factorization of the function*

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B$$

if and only if the operator $\hat{T} := P^\times|\operatorname{Im} P : \operatorname{Im} P \to \operatorname{Im} P^\times$ *is invertible. In this case the factorization is given by* $W(\lambda) = W_-(\lambda)W_+(\lambda)$, *where*

$$W_-(\lambda) = I_Y + CP(\lambda I - PAP)^{-1}\hat{T}^{-1}P^\times B,$$

$$W_+(\lambda) = I_Y + C(I - P^\times)\hat{S}^{\times -1}[\lambda I - (I-P)A(I-P)]^{-1}(I-P)B,$$

$$W_-(\lambda)^{-1} = I_Y - CP\hat{T}^{-1}(\lambda I - P^\times A^\times P^\times)^{-1}P^\times B,$$

$$W_+(\lambda)^{-1} = I_Y - C(I-P^\times)[\lambda I - (I-P^\times)A^\times(I-P^\times)]^{-1}\hat{S}^{\times -1}(I-P^\times)B.$$

Here $\hat{S}^\times = (P - I)|\operatorname{Im}(I-P^\times) : \operatorname{Im}(I-P^\times) \to \operatorname{Im}(I-P)$.

Proof. By the discussion preceding the theorem, the factorization exists if and only if there is a function W_- with c.s.s.d. equal to $\{(C|\operatorname{Im} P, A|\operatorname{Im} P), (A^\times|\operatorname{Im} P^\times, P^\times B), P^\times|\operatorname{Im} P\}$ on $\mathcal{D}$ and $\{(0,0), (0,0), 0\}$ on $\mathcal{D}_e$. For this case the operator $\hat{\mathcal{T}}$ given by (3.1) collapses to $\hat{T} := P^\times|\operatorname{Im} P : \operatorname{Im} P \to \operatorname{Im} P^\times$. We conclude from Theorem 3.1 that such a W_- exists if and only if $\hat{T}$ is invertible. Equations (3.2) and (3.3) for this case then reduce to the formulas for W_- and W_-^{-1} in Theorem 4.1. To get the formulas for W_+ and W_+^{-1}, note that W_+ is determined by

$$W_+^{-1}H_Y^{2\perp} = W^{-1}H_Y^{2\perp}, \qquad W_+^{-1}H_Y^2 = H_Y^2.$$

Thus W_+^{-1} has trivial c.s.s.d. $\{(0,0), (0,0), 0\}$ on $\mathcal{D}$, and a c.s.s.d. on $\mathcal{D}_e$ equal to that of W^{-1}. From $W(\lambda)^{-1} = I_Y - C(\lambda I_X - A^\times)^{-1}B$, use Theorem 1.6 to deduce that a c.s.s.d. for W^{-1} on $\mathcal{D}_e$ is given by

$$\{(-C|\operatorname{Im}(I-P^\times), A^\times|\operatorname{Im}(I-P)),$$
$$(A|\operatorname{Im}(I-P), (I-P)B), (P-I)|\operatorname{Im}(I-P^\times)\}.$$

The operator $\hat{\mathscr{T}}$ given by (3.1) in this case collapses to

$$\hat{S}^{x} := (P - I)|\mathrm{Im}(I - P^{x}) : \mathrm{Im}(I - P^{x}) \to \mathrm{Im}(I - P).$$

We know that $\hat{S}^{x}$ is invertible if and only if such a W_{+}^{-1} exists (by Theorem 3.1), and by the first part of the proof this is the case if and only if $\hat{T} := P^{x}|\mathrm{Im}\, P$ is invertible; in fact it is not difficult to show directly that $\hat{S}^{x}$ is invertible if and only if $\hat{T}$ is invertible. Granting this, we obtain the formulas for W_{+}^{-1} and $W_{+} = (W_{+}^{-1})^{-1}$ as a direct specialization of (3.2) and (3.3) again. ∎

REMARK. In [6] there was obtained the condition

$$\mathrm{Im}\, P \dotplus \mathrm{Im}(I - P^{x}) = X$$

as a criterion for the existence of a canonical Wiener-Hopf factorization by purely state space methods. This condition is easily seen to be equivalent to the invertibility of $P^{x}|\mathrm{Im}\, P : \mathrm{Im}\, P \to \mathrm{Im}\, P^{x}$, the condition in Theorem 4.1. In [7] it was shown, by a completely different "matricial coupling" procedure that the operator $P^{x}|\mathrm{Im}\, P$ is an "indicator" for the Toeplitz operator T_{W}; this also implies that the invertibility of $P^{x}|\mathrm{Im}\, P$ is equivalent to the existence of a canonical Wiener-Hopf factorization.

4.2. *J-Inner-Outer Factorization*

Suppose we are given a rational matrix function $W(\lambda)$ analytic and invertible on the unit circle, at ∞, and at 0 for which we know a minimal realization

$$W(\lambda) = D + C(\lambda I_{X} - A)^{-1}B \tag{4.4}$$

where $D: Y \to Y$, $C: X \to Y$, $A: X \to X$ and $B: Y \to X$. We are also given invertible self-adjoint operators J' and J on Y which are congruent (i.e. J' and J have the same number of positive eigenvalues and the same number of negative eigenvalues). The problem is to decide whether $W(\lambda)$ admits a factorization

$$W(\lambda) = \Theta(\lambda)F(\lambda) \tag{4.5}$$

where Θ and F are rational matrix functions such that

$$\Theta^{*}(\lambda)J'\Theta(\lambda) = J \tag{4.6a}$$

and

$$F(\lambda) \text{ is analytic and invertible on } \bar{\mathscr{D}} \text{ for all } \lambda. \tag{4.6b}$$

Here $\Theta^*(\lambda) := \Theta(\bar{\lambda}^{-1})^*$. As we shall indicate below, this problem has applications to the model reduction problem in systems theory (see [3]).

Suppose now that such a factorization exists. From (4.5) and (4.6b) we see that

$$\Theta H_Y^2 = W H_Y^2.$$

Thus W determines a c.s.s.d. on $\mathscr{D}$ for Θ. But also from (4.5) we get $W^{*-1} = \Theta^{*-1} F^{*-1}$, where F^{*-1} is analytic and invertible on $\mathscr{D}_e$; thus

$$\Theta^{*-1} H_Y^{2\perp} = W^{*-1} H_Y^{2\perp}.$$

From (4.6a), we get $\Theta^{-1} J'^{-1} \Theta^{*-1} = J^{-1}$, so $\Theta^{*-1} = J' \Theta J^{-1}$. Thus

$$\Theta H_Y^{2\perp} = J'^{-1} W^{*-1} H_Y^{2\perp}.$$

In this way we see that we get a c.s.s.d. for Θ on $\mathscr{D}_e$ by computing one for the known function $J'^{-1} W^{*-1}$. We can then solve for Θ (if one exists) by Theorem 3.1. It remains only to normalize the value $\Theta(\infty)$ so that (4.6a) holds. The result is as follows.

THEOREM 4.2. *Let there be given a rational matrix function*

$$W(\lambda) = D + C(\lambda I_X - A)^{-1} B \tag{4.7}$$

analytic and invertible on the unit circle, at 0, *and at* ∞ *along with congruent invertible self-adjoint* J *and* J' *on* Y. *Let* P *and* $P^\times$ *be the Riesz projections for* A *and* $A^\times := A - BD^{-1}C$ *for spectrum in* $\mathscr{D}$. *Then there exists a factorization* $W(\lambda) = \Theta(\lambda)F(\lambda)$ *such that* Θ *satisfies* (4.6a) *and* F *satisfies* (4.6b) *if and only if the operator* $\hat{\mathscr{T}}: \operatorname{Im} P^{\times *} \dot{+} \operatorname{Im} P \to \operatorname{Im} P^* \dot{+} \operatorname{Im} P^\times$ *given by*

$$\hat{\mathscr{T}} = \begin{pmatrix} P^* & -\mathscr{O}^* J' \mathscr{O} \\ \mathscr{C}^\times J'^{-1} \mathscr{C}^{\times *} & P^\times \end{pmatrix}, \tag{4.8}$$

where

$$\mathscr{O} = \operatorname{col}(CA^{j-1}P)_{j=1}^{\infty}$$

and

$$\mathscr{C}^{\times} = \operatorname{row}(P^{\times}A^{\times j-1}BD^{-1})_{j=1}^{\infty},$$

is invertible. In this case any matrix function $\Theta(\lambda)$ *which is the left factor in such a factorization has the form*

$$\Theta(\lambda) = E + \begin{pmatrix} C_- & C_+ \end{pmatrix} \begin{pmatrix} (\lambda I - A_{p-})^{-1} & 0 \\ 0 & (\lambda I - A_{p+})^{-1} \end{pmatrix} \hat{\mathscr{T}}^{-1} \begin{pmatrix} B_- \\ B_+ \end{pmatrix} E$$

where $\hat{\mathscr{T}}$ *is given by* (4.8), *where*

$$C_- = J'^{-1}D^{-1*}B^*A^{\times * -1}|\operatorname{Im} P^{\times *},$$

$$C_+ = C|\operatorname{Im} P,$$

$$A_{p-} = A^{\times * -1}|\operatorname{Im} P^{\times *},$$

$$A_{p+} = A|\operatorname{Im} P,$$

$$B_- = P^*A^{*-1}C^*J',$$

$$B_+ = P^{\times}BD^{-1},$$

and where E *is any operator on* Y *satisfying*

$$E^{*-1}JE^{-1} = J' - J'\begin{pmatrix} C_- & C_+ \end{pmatrix} \begin{pmatrix} A_{p-}^{-1} & 0 \\ 0 & A_{p+}^{-1} \end{pmatrix} \hat{\mathscr{T}}^{-1} \begin{pmatrix} B_- \\ B_+ \end{pmatrix}.$$

Proof. By Theorem 1.3 a c.s.s.d. on $\mathscr{D}$ for $W(\lambda)D^{-1}$ is given by $\{(C|\operatorname{Im} P, A|\operatorname{Im} P), (A^{\times}|\operatorname{Im} P^{\times}, P^{\times}BD^{-1}), P^{\times}|\operatorname{Im} P\}$. Let us compute a c.s.s.d. on $\mathscr{D}_e$ for the function $J'^{-1}W^{*-1}(\lambda)D^* := J'^{-1}W(\bar{\lambda}^{-1})^{*-1}D^*$.

From $W(\lambda)D^{-1} = I_Y + C(\lambda I_X - A)^{-1}BD^{-1}$ we see that

$$DW^{-1}(\lambda) = I_Y - C(\lambda I_X - A^{\times})^{-1}BD^{-1}$$

where $A^{\times} := A - BD^{-1}C$. Thus

$$W(\lambda)^{*-1}D^* = I_Y - D^{-1*}B^*(\bar{\lambda}I_X - A^{\times *})^{-1}C^*$$

and

$$W(\bar{\lambda}^{-1})^{*-1}D^* = I_Y - D^{-1*}B^*\left(\lambda^{-1}I_X - A^{\times*}\right)^{-1}C^*.$$

We compute

$$\begin{aligned}\left(\lambda^{-1}I_X - A^{\times*}\right)^{-1} &= -\lambda A^{\times*-1}\left(\lambda I_X - A^{\times*-1}\right)^{-1}\\ &= -A^{\times*-1} - A^{\times*-2}\left(\lambda I_X - A^{\times*-1}\right)^{-1}.\end{aligned}$$

Thus

$$\begin{aligned}W(\bar{\lambda}^{-1})^{*-1}D^* = {}&\left[I_Y + D^{-1*}B^*A^{\times*-1}C^*\right]\\ &+ D^{-1*}B^*A^{\times*-2}\left(\lambda I_X - A^{\times*-1}\right)^{-1}C^*.\end{aligned}$$

Thus

$$\begin{aligned}&J'^{-1}W^{*-1}D^*\Gamma^{-1}J'\\ &\quad = I_Y + J'^{-1}D^{-1*}B^*A^{\times*-1}\left(\lambda I_X - A^{\times*-1}\right)^{-1}A^{\times*-1}C^*\Gamma^{-1}J'\end{aligned}$$

where $\Gamma := I_Y + D^{-1*}B^*A^{\times*-1}C^*$. From $\Gamma = W(0)^{*-1}D^*$ we compute

$$\Gamma^{-1} = \left[W(0)D^{-1}\right]^* = I_Y - D^{-1*}B^*A^{*-1}C^*.$$

Thus

$$\begin{aligned}A^{\times*-1}C^*\Gamma^{-1} &= A^{\times*-1}C^*\left(I_Y - D^{*-1}B^*A^{*-1}C^*\right)\\ &= A^{\times*-1}\left(I_Y - C^*D^{*-1}B^*A^{*-1}\right)C^*\\ &= A^{\times*-1}\left(A^* - C^*D^{*-1}B^*\right)A^{*-1}C^* = A^{*-1}C^*.\end{aligned}$$

Thus we have the realization of $J'^{-1}W^{*-1}(\lambda)D^*\Gamma^{-1}J'$ given by

$$\begin{aligned}&J'^{-1}W^{*-1}(\lambda)D^*\Gamma^{-1}J'\\ &\quad = I_Y + J'^{-1}D^{*-1}B^*A^{\times*-1}\left(\lambda I_X - A^{\times*-1}\right)^{-1}A^{*-1}C^*J'.\end{aligned}$$

We also need to compute

$$\underline{A}^{\times} := A^{\times * -1} - (A^{* -1}C^{*}J')(J'^{-1}D^{* -1}B^{*}A^{\times * -1})$$
$$= A^{* -1}[A^{*} - C^{*}D^{* -1}B^{*}]A^{\times * -1} = A^{* -1}.$$

Note that the Riesz projection for $A^{\times * -1}$ for spectrum in $\mathscr{D}_e$ is $P^{\times *}$ and the one for $A^{* -1}$ for spectrum in $\mathscr{D}_e$ is P^{*}. Then by Theorem 1.6, a c.s.s.d. on $\mathscr{D}_e$ for $J'^{-1}W^{* -1}$ is given by

$$\{(J'^{-1}D^{* -1}B^{*}A^{\times * -1}|\operatorname{Im} P^{\times *}, A^{\times * -1}|\operatorname{Im} P^{\times *}),$$
$$(A^{* -1}|\operatorname{Im} P^{*}, P^{*}A^{* -1}C^{*}J'), -P^{*}|\operatorname{Im} P^{\times *}\}.$$

If we plug these data into Theorem 3.1 and use the direct analysis of Theorem 1.2 and 1.5, we see that there is a rational matrix function $\Theta'(\lambda)$ such that

$$\Theta' H_Y^2 = W H_Y^2, \qquad \Theta' H_Y^{2\perp} = J'^{-1}W^{* -1}H_Y^{2\perp} \tag{4.9}$$

if and only if the operator $\hat{\mathscr{T}}$ given by (4.8) is invertible. By the discussion preceding the theorem, we see that the invertibility of $\hat{\mathscr{T}}$ as given in (4.8) is a necessary condition for the existence of a factorization (4.5) with (4.6a) and (4.6b).

Conversely, suppose $\hat{\mathscr{T}}$ as in (4.8) is invertible, so there exists a Θ' satisfying (4.9) with $\Theta'(\infty) = I_Y$. Take the J'-orthogonal complement to both identities in (4.9) to get

$$J'^{-1}\Theta'^{* -1}H_Y^{2\perp} = J'^{-1}W^{* -1}H_Y^{2\perp},$$
$$J'^{-1}\Theta'^{* -1}H_Y^{2} = WH_Y^{2}.$$

By the uniqueness statement in Theorem 3.1, conclude that $\Theta' = J'^{-1}\Theta'^{* -1}\hat{J}$ for some invertible constant matrix $\hat{J}$. Thus

$$\Theta'^{*}(\lambda)J'\Theta'(\lambda) = \hat{J} \tag{4.10}$$

for all λ. By choosing $|\lambda| = 1$, we see that $\hat{J}$ is congruent to J as well, and hence has a factorization

$$\hat{J} = E^{* -1}JE^{-1},$$

where E is a constant matrix. Then $\Theta(\lambda) := \Theta'(\lambda)E$ satisfies

$$\Theta H_Y^2 = W H_Y^2, \qquad \Theta H_Y^{2\perp} = J'^{-1} W^{*-1} H_Y^{2\perp} \tag{4.11}$$

and also (4.6a). But it is easy to see that (4.11) is equivalent to $F := \Theta^{-1}W$ satisfying (4.6b). Then $W = \Theta F$ is the desired factorization.

To compute Θ, first use (3.2) with the data as described above to compute Θ'. To compute E, plug $\lambda = 0$ in (4.10) to get $\Theta'(\infty)^* J' \Theta'(0) = \hat{J}$, i.e. $J'\Theta(0) = \hat{J} = E^{*-1} J E^{-1}$. This gives the formula for $\Theta(\lambda)$ in the theorem. ∎

REMARK 4.3. Theorem 4.2 can be applied to the model reduction problem for discrete time systems discussed in [3]. In this problem we are given a stable $p \times q$ matrix function $G(\lambda) = C(\lambda I_n - A)^{-1}B$ where C is $p \times n$, A is $n \times n$, and B is $n \times q$, and $\sigma(A) \subset \mathcal{D}$. We are given a tolerance level σ such that $\sigma_{l+1}(G) < \sigma < \sigma_l(G)$, where $\sigma_j(G)$ $(1 \leqslant j \leqslant n)$ are the Hankel singular values of G (see [3] for details). The problem is to characterize all functions of the form $\hat{G}(\lambda) + F(\lambda)$, where $\hat{G}$ is rational of McMillan degree at most l and $F \in H_{p\times q}^{\infty}$, such that $\|G - \hat{G} - F\|_\infty \leqslant \sigma$ [where $\|H\|_\infty := \sup\{\|H(\lambda)\| \mid |\lambda| = 1\}$]. It is known that there is a matrix function

$$\Theta(\lambda) = \begin{pmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{pmatrix}$$

such that all possible error functions are characterized by

$$G - \hat{G} - F = (\theta_{11}H + \theta_{12})(\theta_{21}H + \theta_{22})^{-1}, \tag{4.12}$$

where H is an arbitrary function in $H_{p\times q}^{\infty}$ with $\|H\|_\infty \leqslant 1$. One of the main results from [1] is that the function Θ for this linear fractional map in (4.12) arises as any rational $(p+q)\times(p+q)$ matrix function Θ satisfying

$$\Theta^*(\lambda)\begin{pmatrix} I_p & 0 \\ 0 & -\sigma^2 I_q \end{pmatrix}\Theta(\lambda) = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{4.13}$$

and

$$\Theta H_{p+q}^2 = W H_{p+q}^2, \tag{4.14}$$

where

$$W(\lambda) = \begin{pmatrix} I_p & G(\lambda) \\ 0 & I_q \end{pmatrix}.$$

One easily checks that the conditions (4.13) and (4.14) are equivalent to $W = \Theta F$ (where $F := \Theta^{-1}W$) being a factorization as in (4.4), (4.5), (4.6) with

$$J' = \begin{pmatrix} I_p & 0 \\ 0 & -\sigma^2 I_q \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Thus the matrix function Θ can be computed directly by applying Theorem 4.2 to

$$W(\lambda) = I_{p+q} + \begin{pmatrix} C \\ 0 \end{pmatrix}(\lambda I_n - A)^{-1}(0, B).$$

The operator $\hat{\mathscr{T}}$ given by (4.8) in this case reduces to the following $2n \times 2n$ matrix acting on $\mathbb{C}^{2n}$:

$$\hat{\mathscr{T}} = \begin{pmatrix} I_n & -\hat{Q} \\ -\sigma^{-2}\hat{P} & I_n \end{pmatrix}$$

where $\hat{P} := \sum_{j=1}^{\infty} A^{j-1}BB^*A^{*\,j-1}$ is the controllability gramian and $\hat{Q} := \sum_{j=1}^{\infty} A^{*\,j-1}C^*CA^{j-1}$ is the observability gramian. Invertibility of $\hat{\mathscr{T}}$ is equivalent to invertibility of $I_n - \sigma^{-2}\hat{Q}\hat{P}$, which is guaranteed by the assumption that $\sigma_{l+1}(G) < \sigma < \sigma_l(G)$. Then the inverse $\hat{\mathscr{T}}^{-1}$ can be computed explicitly as

$$\hat{\mathscr{T}}^{-1} = \begin{pmatrix} A^*Z & A^*Z\hat{Q} \\ \sigma^{-2}\hat{P}Z & Z^* \end{pmatrix}$$

where $Z := (I_n - \sigma^{-2}\hat{Q}\hat{P})^{-1}$. The formula for Θ in Theorem 4.2 for this case reduces to

$$\Theta(\lambda) = \Theta'(\lambda)E$$

where

$$\Theta'(\lambda) = \begin{pmatrix} \theta'_{11}(\lambda) & \theta'_{12}(\lambda) \\ \theta'_{21}(\lambda) & \theta'_{22}(\lambda) \end{pmatrix}$$

is given by

$$\theta_{11}'(\lambda) = I_p + \sigma^{-2}C(\lambda I_n - A)^{-1}\hat{P}ZA^{*-1}C^*,$$

$$\theta_{12}'(\lambda) = C(\lambda I_n - A)^{-1}Z^*B,$$

$$\theta_{21}'(\lambda) = -\sigma^{-2}B^*A^{*-1}(\lambda I_n - A^{*-1})^{-1}ZA^{*-1}C^*,$$

$$\theta_{22}'(\lambda) = I_q - \sigma^{-2}B^*A^{*-1}(\lambda I_n - A^{*-1})^{-1}Z\hat{Q}B,$$

and E is any $(p+q)\times(p+q)$ matrix such that

$$\begin{pmatrix} I - \sigma^{-2}CA^{-1}\hat{P}ZA^{*-1}C^* & -CA^{-1}Z^*B \\ B^*ZA^{*-1}C^* & -\sigma^{-2}I_q + B^*Z\hat{Q}B \end{pmatrix} = E^{*-1}\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}E.$$

These formulas agree with those found in [3], where the outer factor F was computed using methods from [6], and then Θ derived via $\Theta = WF^{-1}$.

The original result of this type for the model reduction problem was obtained by K. Glover [8] for a continuous time setting and for the more difficult case where $\sigma_{l+1}(G) = \sigma$. The analogue of the above result was obtained in [4] for the continuous time setting via the factorization approach. We hope to develop the inverse spectral theory approach of the present paper to a continuous time setting in a future publication.

REMARK 4.4. If $J = J' = I_Y$ in Theorem 4.2 it is known (as a consequence of the Beurling-Lax theorem for example) that the factorization (4.5)–(4.6) of W exists. Thus the operator $\hat{\mathcal{T}}$ given by (4.8) is automatically invertible if $J' = I_Y$. To see this directly, use the identity (3.6) to see that the invertibility of $\hat{\mathcal{T}}$ is equivalent to the invertibility of the operator $\mathcal{T}$ associated with the invariant subspaces M and $M^{x} := M^{\perp}$. Check that in this case $\mathcal{T}$ collapses to an operator of the form

$$\mathcal{T} = \begin{pmatrix} -T^* & I \\ I & T \end{pmatrix}$$

which is easily seen to be invertible.

4.3. Coprime Factorization

As our last application we consider the problem of computing a left coprime factorization $W(\lambda) = D_L^{-1}(\lambda)N_L(\lambda)$ for a known rational matrix function $W(\lambda)$. Here we demand that both $D_L(\lambda)$ and $N_L(\lambda)$ be in $H^{\infty}_{\mathcal{L}(Y)}$ and that they have no nontrivial common left factor in $H^{\infty}_{\mathcal{L}(Y)}$. Equivalently, $D_L(\lambda)$ and $N_L(\lambda)$ are in $H^{\infty}_{\mathcal{L}(Y)}$ and there exist matrix functions $X_L(\lambda)$ and $Y_L(\lambda)$ in $H^{\infty}_{\mathcal{L}(Y)}$ such that the Bezout identity

$$N_L(\lambda)X_L(\lambda) + D_L(\lambda)Y_L(\lambda) = I_Y$$

is satisfied.

Suppose $W(\lambda) = D_L^{-1}(\lambda)N_L(\lambda)$ is a left coprime factorization. From the "no common left factor" criterion, it is not difficult to see that then

$$D_L^{-1}H_Y^2 = P_{H_Y^{2\perp}}\left(WH_Y^2\right) \oplus H_Y^2, \tag{4.15}$$

and conversely, if (4.15) holds, then $W = D_L^{-1}D_LW$ is a left coprime factorization. Then a c.s.s.d. on $\mathcal{D}$ for D_L^{-1} is $\{(C_+, A_{p+}), (0,0), 0\}$ if $\{(C_+, A_{p+}), (A_{z+}, B_+), \hat{T}\}$ is a c.s.s.d. on $\mathcal{D}$ for W. As a matter of normalization we also demand that

$$D_L^{-1}H_Y^{2\perp} = \left(D_L^{-1}H_Y^2\right)^{\perp}. \tag{4.16}$$

From this one can determine a c.s.s.d. on $\mathcal{D}_e$ for D_L^{-1} and then solve for D_L^{-1} (and hence also for D_L) by using Theorem 3.1. One can then determine N_L by $N_L = D_LW$. The result is as follows.

THEOREM 4.5. *Suppose that we are given a rational matrix function*

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B$$

which is analytic and invertible on the unit circle and at 0, *and assume this realization is minimal. Let P be the Riesz projection for A corresponding to spectrum in $\mathcal{D}$. Then we have the left coprime factorization*

$$W(\lambda) = D_L(\lambda)^{-1}N_L(\lambda),$$

where

$$D_L(\lambda) = I_Y + C\hat{Q}^{-1}\left(\lambda I_{\operatorname{Im} P^*} - A^{*-1}P^*\right)^{-1}A^{*-1}P^*C^*,$$

$$D_L(\lambda)^{-1} = I_Y - C(\lambda I_{\operatorname{Im} P} - AP)^{-1}\hat{Q}^{-1}A^{*-1}P^*C^*,$$

$$N_L(\lambda)^{-1} = I_Y - C(\lambda I_X - A^x)^{-1}\left(P\hat{Q}^{-1}A^{*-1}P^*C^* + B\right),$$

and

$$N_L(\lambda) = I_Y + \underline{C}\left(\lambda I_{\operatorname{Im} P^* \dotplus \operatorname{Im}(I-P)} - \underline{A}\right)^{-1}\underline{B},$$

where

$$\underline{A} = \begin{pmatrix} A^{*-1}P^* & A^{*-1}P^*C^*C(I-P) \\ 0 & A(I-P) \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} A^{*-1}P^*C^* + \hat{Q}PB \\ (I-P)B \end{pmatrix}, \qquad \underline{C} = \begin{pmatrix} C\hat{Q}^{-1}P^* & C(I-P) \end{pmatrix}$$

and $\hat{Q}$ is defined by

$$\hat{Q} = \sum_{j=1}^{\infty} P^*A^{*\,j-1}C^*CA^{j-1}P : \operatorname{Im} P \to \operatorname{Im} P^*. \tag{4.17}$$

Proof. Consider the left coprime factorization $W(\lambda) = D_L(\lambda)^{-1}N_L(\lambda)$. From (4.15) and Theorem 1.3 we see that $\{(C|\operatorname{Im} P, A|\operatorname{Im} P), (0,0), 0\}$ is a c.s.s.d. on $\mathscr{D}$ for D_L^{-1} and $\mathscr{F}(D_L^{-1}H_Y^2) = \operatorname{Im}\operatorname{col}(CA^{j-1}P)_{j=1}^{\infty} \oplus l_Y^2$. Thus

$$\mathscr{F}\left(D_L^{-1}H_Y^2\right)^{\perp} = \operatorname{Ker}\operatorname{row}\left(P^*A^{*\,j-1}C^*\right)_{j=1}^{\infty} \oplus (0).$$

From (4.16) and using Theorem 1.5, we read off that a c.s.s.d. on $\mathscr{D}_e$ for D_L^{-1} is $\{(0,0), (A^{*-1}|\operatorname{Im} P^*, A^{*-1}P^*C^*), 0\}$. We now apply Theorem 3.1 to this collection of data. The operator $\hat{\mathscr{T}}$ given by (3.1) in this case collapses to $-\hat{Q}$, where $\hat{Q}: \operatorname{Im} P \to \operatorname{Im} P^*$ is the "observability gramian" given by (4.17). By hypothesis (C, A) is observable, so $\hat{Q}$ is invertible. If we apply the formulas (3.3) and (3.2) as in Theorem 3.1 for this inverse spectral problem for D_L^{-1}, we get the formulas for D_L and D_L^{-1}.

Next, we compute N_L from $N_L = D_L W$. By cascade connection of the realizations for D_L and W we obtain

$$N_L(\lambda) = I_Y + \begin{pmatrix} C\hat{Q}^{-1} & CP & C(I-P) \end{pmatrix}$$
$$\times \left[\lambda - \begin{pmatrix} A^{*-1}P^* & A^{*-1}P^*C^*CP & A^{*-1}P^*C^*C(I-P) \\ 0 & AP & 0 \\ 0 & 0 & A(I-P) \end{pmatrix} \right]^{-1}$$
$$\times \begin{pmatrix} A^{*-1}P^*C^* \\ PB \\ (I-P)B \end{pmatrix}$$
$$= I + C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}A^{*-1}P^*C^*$$
$$+ C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)A^{*-1}P^*C^*CP(\lambda - AP)^{-1}PB$$
$$+ C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}A^{*-1}P^*C^*C$$
$$\times (I-P)[\lambda - A(I-P)]^{-1}(I-P)B$$
$$+ CP(\lambda - AP)^{-1}PB + C(I-P)[\lambda - A(I-P)]^{-1}(I-P)B.$$

Consider the two terms containing $(\lambda - AP)^{-1}$. Using the fact that $\hat{Q}$ solves the Lyapunov equation $\hat{Q} - P^*A^*\hat{Q}AP = P^*C^*CP$, we obtain that the sum of those two terms equals

$$C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}A^{*-1}(\hat{Q} - A^*P^*\hat{Q}AP)(\lambda - AP)^{-1}PB$$
$$+ CP(\lambda - AP)^{-1}PB$$
$$= C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}(A^{*-1}P^* - \hat{Q}AP\hat{Q}^{-1})$$
$$\times (\lambda - \hat{Q}AP\hat{Q}^{-1})^{-1}\hat{Q}PB$$
$$+ CP(\lambda - AP)^{-1}PB.$$

Using the resolvent identity, one sees that this equals $C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}\hat{Q}PB$. A simple grouping together of terms provides the formula for $N_L(\lambda)$.

From the formula for $N_L(\lambda)$, we have

$$N_L(\lambda)^{-1} = I_Y - \underline{C}(\lambda - \underline{A} + \underline{B}\underline{C})^{-1}\underline{B}.$$

Writing out $\underline{A} - \underline{B}\underline{C}$, one obtains

$$\underline{A} - \underline{B}\underline{C} = \begin{pmatrix} A^{*-1}P^* - A^{*-1}P^*C^*C\hat{Q}^{-1}P^* - \hat{Q}PBC\hat{Q}^{-1}P^* & -\hat{Q}PBC(I-P) \\ -(I-P)BCP\hat{Q}^{-1}P^* & (I-P)A^x(I-P) \end{pmatrix}.$$

Using again $P^*C^*CP = \hat{Q} - P^*A^*\hat{Q}AP$, this equals

$$\begin{pmatrix} \hat{Q}PA^xP\hat{Q}^{-1} & \hat{Q}PA^x(I-P) \\ (I-P)A^xP\hat{Q}^{-1} & (I-P)A^x(I-P) \end{pmatrix} = TA^xT^{-1},$$

where $A^x = A - BC$ and $T: X \to \operatorname{Im} P^* \dot{+} \operatorname{Im}(I-P)$ is given by

$$T = \begin{pmatrix} \hat{Q} & 0 \\ 0 & I_{\operatorname{Im}(I-P)} \end{pmatrix}.$$

Then

$$\begin{aligned} N_L(\lambda)^{-1} &= I_Y - \underline{C}T(\lambda - A^x)^{-1}T^{-1}\underline{B} \\ &= I_Y - C(\lambda - A^x)^{-1}(P\hat{Q}^{-1}A^{*-1}P^*C^* + B). \end{aligned}$$ ■

The next lemma will be useful in the determination of formulas for the functions $X_L(\lambda)$ and $Y_L(\lambda)$ which appear in the Bezout identity.

LEMMA 4.6. *Let $A: X \to X$, $B: Y \to X$, and suppose (A, B) is controllable. Let P be the Riesz projection of A corresponding to spectrum in $\mathscr{D}$. Then there exists a matrix $K: X \to Y$ such that the following properties hold:*

(i) $KP = K$,
(ii) $(I-P)BK = 0$,
(iii) *the spectrum of $A - BK$ lies in $\mathscr{D}_e$.*

Proof. Since (PAP, PB) is controllable, there is a $\hat{K}: \operatorname{Im} P \to Y$ such that the spectrum of $PAP - PB\hat{K}$ lies in $\mathscr{D}_e$. We can view $\tilde{K} = \hat{K}P$ as an operator mapping $X \to Y$, so (i) and (iii) hold. Now decompose

$$Y = B^{-1} \operatorname{Im} P|_{\operatorname{Ker} B} \dot{+} \operatorname{Ker} B \dot{+} B^{-1}\operatorname{Im}(I-P)|_{\operatorname{Ker} B},$$

and write with respect to this decomposition $\tilde{K}x = K_1x + K_2x + K_3x$. Then $B\tilde{K}x = BK_1x + BK_3x$, and $PB\tilde{K}x = BK_1x$, $(I-P)B\tilde{K}x = BK_3x$. Since with respect to the decomposition $X = \operatorname{Im} P \dot{+} \operatorname{Im}(I-P)$ we have

$$A - B\tilde{K} = \begin{pmatrix} PAP - PB\tilde{K} & 0 \\ (P-I)B\tilde{K}P & A(I-P) \end{pmatrix} = \begin{pmatrix} PAP - BK_1 & 0 \\ -BK_3 & A(I-P) \end{pmatrix},$$

it is clear that the spectrum of $A - BK$, where $Kx = K_1x$, lies in $\mathscr{D}_e$. So by taking $K = K_1$ we obtain the lemma. ∎

We now give formulas for the functions $X_L(\lambda)$ and $Y_L(\lambda)$ appearing in the Bezout identity. The existence of a matrix K as in Lemma 4.6 will be used.

THEOREM 4.7. *Let K be any matrix such that $KP = K$, $(I-P)BK = 0$, and $A - BK$ has all its spectrum lying in $\mathscr{D}_e$. Put*

$$X_L(\lambda) = -K(\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^*,$$

$$Y_L(\lambda) = I - (CP - K)(\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^*.$$

Then

$$N_L(\lambda)X_L(\lambda) + D_L(\lambda)Y_L(\lambda) = I.$$

Proof. From Theorem 4.5 we obtain

$$\begin{aligned}(N_L \quad D_L) &= (I \quad I) + \left(C\hat{Q}^{-1}P^* \quad C(I-P)\right) \\ &\quad\times\left[\lambda - \begin{pmatrix} A^{*-1}P^* & A^{*-1}P^*C^*C(I-P) \\ 0 & A(I-P)\end{pmatrix}\right]^{-1} \\ &\quad\times\begin{pmatrix} A^{*-1}P^*C^* + \hat{Q}PB & A^{*-1}P^*C^* \\ (I-P)B & 0\end{pmatrix}.\end{aligned}$$

Also we can write

$$\begin{pmatrix} X_L(\lambda) \\ Y_L(\lambda)\end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} - \begin{pmatrix} K \\ CP - K\end{pmatrix}(\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^*.$$

Multiplying out, we get

$$\begin{aligned}
&\begin{pmatrix} N_L & D_L \end{pmatrix}\begin{pmatrix} X_L \\ Y_L \end{pmatrix} \\
&\quad = I + C\hat{Q}^{-1}(\lambda - A^{*-1}P^*)^{-1}A^{*-1}P^*C^* \\
&\quad - CP(\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^* \\
&\quad - \begin{pmatrix} C\hat{A}^{-1}P^*C(I-P) \end{pmatrix}\left[\lambda - \begin{pmatrix} A^{*-1}P^* & A^{*-1}P^*C^*C(I-P) \\ 0 & A(I-P) \end{pmatrix}\right]^{-1} \\
&\quad \times \begin{pmatrix} \hat{Q}PBK + A^{*-1}P^*C^*CP \\ (I-P)BK \end{pmatrix}(\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^*.
\end{aligned}$$

Since $(I-P)BK = 0$ the last term equals

$$\begin{aligned}
&- C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}(\hat{Q}PBK + A^{*-1}P^*C^*CP) \\
&\quad \times (\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^*.
\end{aligned}$$

Again using the Lyapunov equation $\hat{Q} - P^*A^*\hat{Q}AP = P^*C^*CP$, we have that this is equal to

$$\begin{aligned}
&- C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}[A^{*-1}\hat{Q} - \hat{Q}(AP - BK)] \\
&\quad \times (\lambda - AP + BK)^{-1}\hat{Q}^{-1}A^{*-1}C^* \\
&= - C\hat{Q}^{-1}P^*(\lambda - A^{*-1}P^*)^{-1}[A^{*-1} - \hat{Q}(AP - BK)\hat{Q}^{-1}] \\
&\quad \times (\lambda - \hat{Q}(AP - BK)\hat{Q}^{-1})A^{*-1}C^*.
\end{aligned}$$

Using the resolvent identity, one readily sees that

$$\begin{pmatrix} N_L & D_L \end{pmatrix}\begin{pmatrix} X_L \\ Y_L \end{pmatrix} = I.$$

Moreover it is clear that $X_L(\lambda)$ and $Y_L(\lambda)$ both are in $H^\infty_{\mathscr{L}(Y)}$. ∎

The analysis for right coprime factorizations $W(\lambda) = N_R(\lambda)D_R^{-1}(\lambda)$ is analogous. Here one demands that $N_R(\lambda)$ and $D_R(\lambda)$ in $H^\infty_{\mathscr{L}(Y)}$ have no common nontrivial right factor, or equivalently N_R and D_R in $H^\infty_{\mathscr{L}(Y)}$ are such that there exists a solution pair $X_R(\lambda), Y_R(\lambda)$ in $H^\infty_{\mathscr{L}(Y)}$ of the Bezout identity

$$X_R(\lambda)N_R(\lambda) + Y_R(\lambda)D_R(\lambda) = I_Y.$$

If $W(\lambda) = N_R(\lambda)D_R^{-1}(\lambda)$ is a right coprime factorization, then

$$N_R H^2_Y = W H^2_Y \cap H^2_Y, \tag{4.18}$$

and conversely. As a normalization we also impose

$$N_R H^{2\perp}_Y = (N_R H^2_Y)^\perp. \tag{4.19}$$

Then we may solve for N_R as an application of Theorem 3.1, after which we find D_R from $D_R = W^{-1}N_R$. Also, we can provide formulas for the functions X_R and Y_R. The result is as follows.

THEOREM 4.8. *Suppose we are given a rational matrix function*

$$W(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B,$$

which is analytic and invertible on the unit circle and at 0, *and assume this realization is minimal. Let* P^x *be the Riesz projection for* $A^x = A - BC$ *corresponding to spectrum in* $\mathscr{D}$. *Then we have the right coprime factorization*

$$W(\lambda) = N_R(\lambda)D_R(\lambda),$$

where

$$N_R(\lambda) = I_Y + B^*P^{x*}A^{x*\,-1}(\lambda I - A^{x*\,-1}P^{x*})^{-1}\hat{P}^{-1}P^xB,$$

$$N_R(\lambda)^{-1} = I_Y - B^*P^{x*}A^{x*\,-1}\hat{P}^{-1}(\lambda I - A^xP^x)^{-1}P^xB,$$

$$D_R(\lambda)^{-1} = I_Y + (C - B^*P^{x*}A^{x*\,-1}\hat{P}^{-1}P^x)(\lambda - A)^{-1}B,$$

and

$$D_R(\lambda) = I_Y + \tilde{C}(\lambda - \tilde{A})^{-1}\tilde{B},$$

where

$$\tilde{A} = \begin{pmatrix} A^x(I - P^x) & (I - P^x)BB^*P^{x*}A^{x*\,-1} \\ 0 & A^{x*\,-1}P^{x*} \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} -C(I - P^x) & -C\hat{P} + B^*P^{x*}A^{x*\,-1} \end{pmatrix},$$

$$\tilde{B} = \begin{pmatrix} (I - P^x)B \\ \hat{P}^{-1}P^xB \end{pmatrix}$$

and $\hat{P}: \operatorname{Im} P^{x*} \to \operatorname{Im} P^x$ *is defined by*

$$\hat{P} = \sum_{j=1}^{\infty} P^xA^{x\,j-1}BB^*A^{x*\,j-1}P^{x*}. \tag{4.20}$$

Further, if F is any matrix with $P^xF = F$, $FC(I - P^x) = 0$ and such that the spectrum of $A^xP^x + FCP^x$ lies in $\mathcal{D}_e$, then we can take

$$X_R(\lambda) = I - B^*P^{x*}A^{x*\,-1}\hat{P}^{-1}(\lambda - A^xP^x - FC)^{-1}(P^xB - F),$$

$$Y_R(\lambda) = -B^*P^{x*}A^{x*\,-1}\hat{P}^{-1}(\lambda - A^xP^x - FC)^{-1}F.$$

Proof. If $W(\lambda) = N_R(\lambda)D_R(\lambda)^{-1}$ is a right coprime factorization of W, then by (4.18) and Theorem 1.3 we see that $\{(0,0),(A^x|\operatorname{Im} P^x, P^xB),0\}$ is a c.s.s.d. on $\mathcal{D}$ for N_R. Using (4.19) we get that $\{(B^*A^{x*\,-1}|\operatorname{Im} P^{x*}, A^{x*\,-1}|\operatorname{Im} P^{x*}),(0,0),0\}$ is a c.s.s.d. on $\mathcal{D}_e$ for N_R. With these data the operator $\hat{\mathcal{T}}$ given by (3.1) collapses to the "controllability gramian" $\hat{P}: \operatorname{Im} P^{x*} \to \operatorname{Im} P^x$ given by (4.20). By the hypothesis that (A, B) is controllable, we see that P is invertible. Finally when we apply (3.2) and (3.3) to this inverse spectral problem for N_R, we get the formulas for N_R and N_R^{-1}.

Next we compute $D_R = W^{-1}N_R$ by a cascade connection of the realizations for W^{-1} and N_R. One obtains

$$
\begin{aligned}
D_R(\lambda) &= W(\lambda)^{-1}N_R(\lambda)\\
&= I + \begin{pmatrix} -CP^x & -C(I-P^x) & B^*P^{x*}A^{x*-1}\end{pmatrix}\\
&\quad \times \left[\lambda - \begin{pmatrix} A^xP^x & 0 & P^xBB^*P^{x*}A^{x*-1}\\ 0 & A^x(I-P^x) & (I-P^x)BB^*P^{x*}A^{x*-1}\\ 0 & 0 & A^{x*-1}P^{x*}\end{pmatrix}\right]^{-1}\\
&\quad \times \begin{pmatrix} P^xB\\ (I-P^x)B\\ \hat{P}^{-1}P^xB\end{pmatrix}\\
&= I - CP^x(\lambda - A^xP^x)^{-1}P^xB - C(I-P^x)[\lambda - A^x(I-P^x)]^{-1}(I-P^x)B\\
&\quad + B^*P^{x*}A^{x*-1}(\lambda - A^{x*-1}P^{x*})^{-1}\hat{P}^{-1}P^xB\\
&\quad - CP^x(\lambda - A^xP^x)^{-1}P^xBB^*P^{x*}A^{x*-1}(\lambda - A^{x*-1}P^{x*})^{-1}\hat{P}^{-1}P^xB\\
&\quad - C(I-P^x)[\lambda - A^x(I-P^x)]^{-1}(I-P^x)\\
&\quad \times BB^*P^{x*}A^{x*-1}(\lambda - A^{x*-1}P^{x*})^{-1}\hat{P}^{-1}P^xB.
\end{aligned}
$$

Take together the terms involving $(\lambda - A^xP^x)^{-1}$, use the fact that $P^xBB^*P^{x*} = \hat{P} - P^xA^x\hat{P}A^{x*}P^{x*}$ in the double product involving $(\lambda - A^xP^x)^{-1}$, and use the resolvent identity as in the proof of Theorem 4.5 to obtain that $D_R(\lambda) = I + \tilde{C}(\lambda - \tilde{A})^{-1}\tilde{B}$ as desired. The computation of $D_R(\lambda)^{-1}$ involves noting that

$$
\tilde{A} - \tilde{B}\tilde{C} = \begin{pmatrix} I_{\mathrm{Im}(I-P^x)} & 0\\ 0 & \hat{P}^{-1}\end{pmatrix} A \begin{pmatrix} I_{\mathrm{Im}(I-P^x)} & 0\\ 0 & \hat{P}\end{pmatrix}.
$$

Using this and $D_R(\lambda)^{-1} = I - \tilde{C}(\lambda - \tilde{A} + \tilde{B}\tilde{C})^{-1}\tilde{B}$ yields the formula for $D_R(\lambda)^{-1}$.

The existence of an operator $F: Y \to X$ such that $P^xF = F$, $FC(I-P^x) = 0$, and the spectrum of $A^xP^x + FCP^x$ lies in $\mathcal{D}_e$ is proved analogously to Lemma 4.6.

To show that the Bezout identity holds with X_R and Y_R as in the theorem, write

$$\begin{pmatrix} N_R(\lambda) \\ D_R(\lambda) \end{pmatrix} = \begin{pmatrix} I \\ I \end{pmatrix} + \begin{pmatrix} 0 & B^*P^{\times *}A^{\times * -1} \\ -C(I-P^{\times}) & -C\hat{P} + B^*P^{\times *}A^{\times * -1} \end{pmatrix}$$
$$\times \left[\lambda - \begin{pmatrix} A^{\times}(I-P^{\times}) & (I-P^{\times})BB^*P^{\times *}A^{\times * -1} \\ 0 & A^{\times * -1}P^{\times *} \end{pmatrix} \right]^{-1}$$
$$\times \begin{pmatrix} (I-P^{\times})B \\ \hat{P}^{-1}P^{\times}B \end{pmatrix}$$

and likewise

$$\big(X_R(\lambda) \quad Y_R(\lambda) \big) = (I \quad 0) - B^*P^{\times *}A^{\times * -1}\hat{P}^{-1}(\lambda - A^{\times}P^{\times} - FC)^{-1}(P^{\times}B - F$$

Multiplying out, we get

$$\big(X_R(\lambda) \quad Y_R(\lambda) \big) \begin{pmatrix} N_R(\lambda) \\ D_R(\lambda) \end{pmatrix}$$
$$= I - B^*P^{\times *}A^{\times * -1}\hat{P}^{-1}(\lambda - A^{\times}P^{\times} - FC)^{-1}P^{\times}B$$
$$+ B^*P^{\times *}A^{\times * -1}(\lambda - A^{\times * -1}P^{\times *})^{-1}\hat{P}^{-1}P^{\times}B$$
$$- B^*P^{\times *}A^{\times * -1}\hat{P}^{-1}(\lambda - A^{\times}P^{\times} - FC)^{-1}$$
$$\times (P^{\times}BB^*P^{\times *}A^{\times * -1} - FC\hat{P})(\lambda - A^{\times * -1}P^{\times *})^{-1}\hat{P}^{-1}P^{\times}B.$$

Using $P^{\times}BB^*P^* = \hat{P} - P^{\times}A^{\times}\hat{P}A^{\times *}P^{\times *}$ in the last term gives that this term equals

$$B^*P^{\times}A^{\times * -1}\hat{P}^{-1}(\lambda - A^{\times}P^{\times} - FC)^{-1}\hat{P}\left[A^{\times * -1} - \hat{P}^{-1}(P^{\times}A^{\times} + FC)\hat{P} \right]$$
$$\times (\lambda - A^{\times * -1}P^{\times *})^{-1}\hat{P}^{-1}P^{\times}B$$
$$= B^*P^{\times *}A^{\times * -1}(\lambda - A^{\times * -1}P^{\times *})^{-1}\hat{P}^{-1}P^{\times}B - B^*P^{\times *}A^{\times * -1}\hat{P}^{-1}$$
$$\times (\lambda - A^{\times}P^{\times} - FC)^{-1}P^{\times}B.$$

The last identity follows from the resolvent identity. This gives the desired result. ■

In [13] coprime factorization and the Bezout identity elements are used to parameterize all stabilizing controllers for $W(s)$. Theorems 4.5, 4.7, and 4.8 allow one to express this parameterization in terms of the operators A, B, C, $\hat{P}$, $\hat{Q}$, K, and F.

For another approach to coprime factorization, giving formulas for all the functions involved in state space terms, see [12]. Indeed, our formulas for the Bezout identity elements were inspired by their results.

Note added in proof. A form of our Theorem 4.1 in the context of Wiener-Hopf integral operators appears in the article "An indicator for Wiener-Hopf integral equations with invertible analytic symbol" (*Integral Equations Operator Theory* 6:1–20 (1983)) by H. Bart and L. G. Kroon. There the inverse of the Wiener-Hopf integral operator is given explicitly in terms of $(P^*|_{\operatorname{Im} P})^{-1}$ by a method that bypasses factorization.

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